Interpolation Sets For Compact Abelian Groups

by

Xu Yang

A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Master of Mathematics
in
Pure Mathematics

Waterloo, Ontario, Canada, 2014

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

In this thesis, we study various properties of I_0 and ε-Kronecker sets and we show that most infinite sets in the discrete dual group contain infinite interpolation sets.

Acknowledgements

Firstly, I thank my advisor Dr. Kathryn Hare for her guidance, patience and encouragement.

I also thank Dr. Andu Nica and Dr. Nico Spronk for reading my thesis.

Finally, I thank the Department of Pure Mathematics and the University of Waterloo for providing me with good study environment and financial support.

Dedication

This is dedicated to my parents.

Contents

Chapter 1

Background and preliminary results

1.1 Introduction

Many classical harmonic analysis results are based on 'Lacunary or Hadamard sets'. Weierstrass used Hadamard sequences to build the first example of nowhere differentiable continuous function. Hadamard sets also inspired the classical Hadamard gap theorem and the Riesz product measure, which is an example of a continuous measure whose Fourier coefficients do not vanish at infinity. Zygmund extensively studied analytic properties of the trigonometric series whose Fourier tranforms were supported on a Hadamard set [21].

Lacunary sets have various interpolation properties. For example, every function to the complex unit circle can be approximated by a continuous character (equivalently, the Fourier Transform of a single point mass measure) within a small error. This inspired the idea of ε-Kronecker sets. The terminology 'ε-Kronecker' was motivated by the classical theorem of Kronecker, which states that every infinite independent subset of the real line is a Kronecker set, meaning the approximation is exact.

An I_0 set, the main topic of this paper, is a weaker interpolation notion. Rather than interpolating with a single continuous character, I_0 sets permit the use of limits of finite linear combinations of Fourier transforms of point mass measures, i.e., they are the Fourier transforms of discrete measures. These interpolation sets were first considered in the 1960's and have been extensively studied since that time. They are special examples of Sidon sets [8]. Early research on I_0 sets was done by mathematicians such as C. Ryll-Nardzewski [20] and S. Hartman [10].

Topologically, I_0 sets are closely related to almost periodic functions, the continuous functions that are periodic within any desired level of accuracy. The original definition of an I_0 set was a set E such that 'every bounded function on $E \subset \mathbb{Z}$ can be extended to an almost periodic function'. But these extensions could always be found in the space consisting of Fourier transforms of discrete measures restricted to E. Thus, I_0 sets are sets such that 'every bounded function is a Fourier transform of discrete measures', the modern definition.

The focus of this thesis is an existence result: infinite ε-Kronecker sets or I_0 sets can be found in most infinite subsets of discrete abelian groups. This is proved in Chapter 3. In Chapter 4, we show an alternative, topological approach to the existence result. We give basic properties of interpolation sets in Chapter 2.

1.2 Background

In this section, we give background harmonic analysis results and examples. For more details and proofs on the background information readers can see [19].

1.2.1 The Haar measure

We begin by introducing the Haar measure of a locally compact abelian group.

For any locally compact abelian group G, there exists a positive, regular, non-zero Borel measure m, called the Haar measure on G, which satisfies the following
(1) m is both left and right invariant. That is, for every $S \subset G$ measurable and

$x \in G$, we have $m(xS) = m(Sx) = m(S)$.
(2) m is inversion invariant. That is, for every $S \subset G$ measurable, we have $m(S^{-1}) = m(S)$.
(3) For any compact $K \subset G$, $m(K) < \infty$.
(4) For any open $U \subset G$, $m(U) > 0$.
Moreover, this Haar measure is unique up to a positive scaling.

Example 1.1. (1) Let $G = \mathbb{T}$, the unit circle group in $\mathbb{C}$, with the usual topology. The Haar measure on G is the (normalized) Lebesgue measure.
(2) Let $G = \mathbb{Z}$ with the discrete topology. The Haar measure on G is the counting measure.

With respect to the Haar measure m and for $p < \infty$, we can define the space $L^p(G)$ to be

$$L^p(G) := \{f : G \to \mathbb{C} \ : \ \int_G |f(x)|^p \ dm(x) < \infty\}/\sim,$$

where the equivalence realtion is given by

$$f \sim g \Leftrightarrow m(\{x \in G : f(x) \neq g(x)\}) = 0.$$

$L^\infty(G)$ is the space of essentially bounded functions quotient the equivalence relation given above.

1.2.2 Dual group and the Fourier transform

We next introduce the construction of the dual group.

For a locally compact abelian group G, we denote by $\widehat{G}$, or Γ, its dual group, defined by

$$\Gamma := \{\gamma \mid \gamma : G \to \mathbb{T}, \gamma \text{ is continuous and multiplicative}\},$$

where the group operation on Γ is pointwise multiplication. Thus, for every $\gamma \in \Gamma$ we have $\gamma(x+y) = \gamma(x)\gamma(y)$ for every $x, y \in G$ and the identity $1 \in \Gamma$, satisfies $1(x) = 1$ for all $x \in G$. Elements in Γ are called the continuous characters on G.

The Fourier transform of a function $f \in L^1(G)$ is the function $\widehat{f} : \Gamma \to \mathbb{C}$ given by $\widehat{f}(\gamma) = \int_G f(x)\overline{\gamma(x)}\ dm(x)$.

We denote by $M(G)$ the Banach space of complex, finite, regular Borel measures on G and for $\mu \in M(G)$, we define the Fourier Stieltjes transform $\widehat{\mu}$ via $\widehat{\mu} : \Gamma \to \mathbb{C}$, where $\widehat{\mu}(\gamma) = \int_G \overline{\gamma(x)}\ d\mu(x)$. We denote by $M_d(G)$ the space of discrete measures, the finite, complex measures concentrated on a countable subset of G. For every $\mu \in M_d(G)$, we have $\mu = \sum_{i=1}^{\infty} a_i \delta_{x_i}$ for some $a_i \in \mathbb{C}$, $x_i \in G$ and δ_{x_i} is the point mass measure at x_i. We denote by $M_d^+(G)$ the set of discrete measures with non-negative coefficients. If, in addition, μ is concentrated on $U \subset G$, we write $\mu \in M_d(U)$ (resp. $M_d^+(U)$).

Let $f, g \in L^1(G)$. The convolution of f and g, denoted by $f * g$, is a function $f * g : G \to \mathbb{C}$, given almost everywhere by

$$f * g(x) = \int_G f(t)g(x-t)\ dm(t).$$

If $f, g \in L^1(G)$, then $f * g \in L^1(G)$ and $f * g = g * f$. Similarly, for $\mu, \nu \in M(G)$ and a Borel set $E \subset G$, we define $E' = \{(x, y) \in G \times G \mid x + y \in E\}$ and define $\mu * \nu(E) = \mu \times \nu(E')$. If $\mu, \nu \in M(G)$, then $\mu * \nu \in M(G)$ and $||\mu * \nu||_{M(G)} \leq ||\mu||_{M(G)}\, ||\nu||_{M(G)}$.

To give a topology on Γ, we use the following identification.

Proposition 1.2. *The map*

$$\Phi : \Gamma \to L^1(G)^*,$$

given by $\Phi(\gamma)(f) = \widehat{f}(\gamma) = \int_G f(x)\overline{\gamma(x)}\ dm(x)$ *for all* $f \in L^1(G), \gamma \in \Gamma$, *is a bijection.*

Through this identification, we can give Γ the weak* topology inherited from $L^1(G)^*$.

Proposition 1.3. *Equipped with topology described above, Γ is again a locally compact abelian group.*

We have the following duality theorem.

Theorem 1.4 (Pontryagin's Duality Theorem)**.** *A locally compact abelian group G is isomorphic to $\widehat{\Gamma}$ as topological groups via the embedding $x \to \widehat{x}$, where $\widehat{x}(\gamma) = \gamma(x)$ for $\gamma \in \Gamma$.*

Hence, we may identify the dual group $\widehat{\Gamma}$ as G. From the duality theorem, we also have the following result about the duality between compact groups and discrete groups.

Proposition 1.5. *An abelian group G is compact if and only if its dual group Γ is discrete.*

Example 1.6. Consider the compact group $\mathbb{T}$. Notice that $\mathbb{T}$ can be identified as $[-\pi, \pi] = \mathbb{R}/2\pi$ with addition mod 2π as the group operation. The continuous characters on $\mathbb{T} = [-\pi, \pi]$ are of the form $\gamma(x) = e^{inx}$ for some $n \in \mathbb{Z}$ and the topology on $\widehat{\mathbb{T}}$ is indeed the discrete topology. Therefore, the dual group $\widehat{\mathbb{T}}$ is the discrete group $\mathbb{Z}$. Moreover, if $\gamma : \mathbb{Z} \to \mathbb{T}$ is a continuous character on $\mathbb{Z}$, put $\gamma(1) = x_\gamma \in \mathbb{T}$ and then we have $\gamma(n) = x_\gamma^n$. Conversely, any function of the form $f(n) = x^n$ for some $x \in \mathbb{T}$ is a continuous character on $\mathbb{Z}$. Hence, $\widehat{\mathbb{Z}} = \mathbb{T}$, which is compact, as expected.

Let $H \subset G$ be a closed subgroup. We denote by $H^\perp$ the annihilator of H. We have the following relation between dual groups and quotient groups.

Proposition 1.7. *$H^\perp = \widehat{G/H}$ and $\widehat{H} = \Gamma/H^\perp$.*

Suppose G_i is a compact abelian group for each $i \in I$. The direct product group $\prod_{i \in I} G_i$ is again a compact abelian group with the product topology.

Proposition 1.8. *Let G_i be compact abelian groups for $i \in I$. We have*

$$\widehat{\prod_{i \in I} G_i} = \bigoplus_{i \in I} \widehat{G_i},$$

where $\bigoplus_{i\in I}\widehat{G_i}$ is the direct sum, meaning each $\oplus_{i\in I}\gamma_i \in \bigoplus_{i\in I}\widehat{G_i}$ has only finitely many coordinates different from the identity.

1.2.3 Bohr compactification

We denote by G_d the group G equipped with the discrete topology. Its compact dual group is $\overline{\Gamma}$, called the Bohr compactifiction of Γ. For a subset $E \subset \Gamma$, we denote its closure in $\overline{\Gamma}$ by $\overline{E}$. It is known that Γ is dense in $\overline{\Gamma}$ and every element in $\overline{\Gamma}$ is a cluster point of Γ.

Notice that if $\mu \in M_d(G)$, then $\widehat{\mu}$ is a continuous function on $\overline{\Gamma}$ because it is the Fourier transform of an ℓ^1 function on G_d.

1.2.4 Other notation

Unless specified otherwise, throughout this thesis, G denotes a compact abelian group and Γ will be its discrete dual group. We denote the identity of G by e and the identity of Γ by 1, except when $G = \mathbb{T}$ and $\Gamma = \mathbb{Z}$, we use 1 and 0 respectively.

Chapter 2

Interpolation sets

We start with definitions.

Let $E \subset \Gamma$ be a subset. We say that a function $\varphi : E \to \mathbb{C}$ is Hermitian if for every $\gamma \in E$ with $\gamma^{-1} \in E$, we have $\varphi(\gamma) = \overline{\varphi(\gamma^{-1})}$.

Definition 2.1. Let $U \subset G$ be a Borel set and $E \subset \Gamma$. We say E is $I_0(U)$ (resp., $FZI_0(U)$) if for every bounded (and Hermitian) $\varphi : E \to \mathbb{C}$ there exists $\mu \in M_d(U)$ (resp., $\mu \in M_d^+(U)$) such that $\widehat{\mu} = \varphi$ on E.
When $U = G$, we just say E is I_0 (resp., FZI_0).

Note that if $\mu = \sum_{i=1}^{\infty} a_i \delta_{x_i}$ is a real discrete measure (ie. each $a_i \in \mathbb{R}$), then for $\gamma \in E$, we have $\widehat{\mu}(\gamma) = \sum_{i=1}^{\infty} a_i \overline{\gamma(x_i)} = \overline{\sum_{i=1}^{\infty} a_i \gamma(x_i)} = \overline{\widehat{\mu}(\gamma^{-1})}$. Thus, the Fourier transform of a real discrete measure is Hermitian and hence the reason for only requiring the interpolation of Hermitian functions in the definition of FZI_0 sets.

Definition 2.2. Let $U \subset G$ be a Borel subset, $E \subset \Gamma$ and $\varepsilon > 0$. We say E is ε-Kronecker(U) if for every function $\varphi : E \to \mathbb{T}$ there exists $x \in U$ such that $|\varphi(\gamma) - \gamma(x)| < \varepsilon$ for all $\gamma \in E$.
We say E is weak ε-Kronecker(U) if for every $\varphi : E \to \mathbb{T}$ there exists $x \in U$ such that $|\varphi(\gamma) - \gamma(x)| \leq \varepsilon$ for all $\gamma \in E$.
When $U = G$, we just say E is ε-Kronecker (resp., weak ε-Kronecker).

In this chapter, we will prove basic properties of I_0, FZI_0 and ε-Kronecker sets. Moreover, we will demonstrate how ε-Kronecker sets are related to the other interpolation sets.

We say that $U \subset G$ is symmetric if $U = U^{-1}$. The next result implies that any $FZI_0(U)$ set is $I_0(U)$ when $U \subset G$ is symmetric.

Proposition 2.3. *Assume $U \subset G$ is symmetric. If $E \subset \Gamma$ is $FZI_0(U)$, then $E \cup E^{-1}$ is $I_0(U)$.*

Notation: For $\mu = \sum_{i=1}^{\infty} a_i \delta_{x_i}$ where $x_i \in U$ and $a_i \in \mathbb{C}$, we define $\widetilde{\mu} = \sum_{i=1}^{\infty} \overline{a_i} \delta_{x_i^{-1}}$. Notice also $\widetilde{\mu} \in M_d(U^{-1})$.

Proof. Suppose E is $FZI_0(U)$. Each $\varphi \in \ell^\infty(E \cup E^{-1})$ can be decomposed as a sum of a Hermitian function and an anti-Hermitian function by $\varphi = \varphi_1 - i\varphi_2$, where $\varphi_1(\gamma) = (\varphi(\gamma) + \overline{\varphi(\gamma^{-1})})/2$ and $\varphi_2(\gamma) = i(\varphi(\gamma) - \overline{\varphi(\gamma^{-1})})/2$ are Hermitian functions. We can find $\mu_1, \mu_2 \in M_d^+(U)$ such that $\widehat{\mu_1} = \varphi_1$ and $\widehat{\mu_2} = \varphi_2$. Then, if we put

$$\nu := \frac{1}{2}(\mu_1 + \widetilde{\mu_1}) - \frac{i}{2}(\mu_2 - \widetilde{\mu_2}),$$

we have that $\widehat{\nu}(\gamma) = \varphi(\gamma)$ for all $\gamma \in E \cup E^{-1}$. So $E \cup E^{-1}$ is $I_0(U)$. $\square$

Remark. In [8], it is shown that if E does not contain 1, then E is FZI_0 if and only if $E \cup E^{-1}$ is I_0.

2.1 Characterizations and examples of interpolation sets

We give a characterization of I_0 (FZI_0) sets.
Notation: For a Banach space X, we denote the unit ball in X by $B(X)$.

Theorem 2.4 ([4], [13], [17]). *Let U be a compact subset of G and $E \subset \Gamma$. The following are equivalent.*
(1) E is $I_0(U)$ (resp,. $FZI_0(U)$).
(2) There exists a constant K such that for every (Hermitian) $\phi \in B(\ell^\infty(E))$, there exists $\mu \in M_d(U)$ (resp,. $M_d^+(U)$) such that $\widehat{\mu}(\gamma) = \phi(\gamma)$ for all $\gamma \in E$ and

$||\mu||_{M(G)} < K$.
(3) There exists some $0 < \varepsilon < 1$ *such that for every (Hermitian)* $\phi \in B(\ell^\infty(E))$, *there exists* $\mu \in M_d(U)$ *(resp.,* $M_d^+(U)$*) with* $||\mu|| \leq K$ *and* $|\widehat{\mu}(\gamma) - \phi(\gamma)| < \varepsilon$ *for all* $\gamma \in E$.
(4) There exist $0 < \varepsilon < 1$ *and an integer* N *such that for every (resp., Hermitian)* $\phi \in B(\ell^\infty(E))$, *there exist* $c_i \in \mathbb{C}$ *(resp.,* $c_i \in \mathbb{R}^+$*) and* $x_i \in U$, $1 \leq i \leq N$, *such that* $\mu = \sum_{i=1}^N c_i \delta_{x_i}$ *satisfies* $|\phi(\gamma) - \widehat{\mu}(\gamma)| < \varepsilon$ *for all* $\gamma \in E$.

Proof. Notice that (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (3) are clear.

We first prove the equivalences among (1), (2) and (3) in the $I_0(U)$ case. The $FZI_0(U)$ case follows similarly.

(1) $\Rightarrow$ (2)
Suppose $E \subset \Gamma$ is $I_0(U)$. Consider the map $T : M_d(U) \to \ell^\infty(E)$ given by $\mu \to \widehat{\mu}|_E$. Since T is linear and for every $\mu \in M_d(U)$, we have

$$||\widehat{\mu}||_{\ell^\infty(E)} = \sup_{\gamma \in \mathcal{E}} |\widehat{\mu}(\gamma)| \leq ||\mu||_{M(G)} \, ,$$

T is continuous. Moreover, since E is $I_0(U)$, T is also surjective. By the Open Mapping Theorem, T is an open map. Consider

$$T' : M_d(U)/\ker(T) \to \ell^\infty(E)$$

such that $T = T' \circ q$, where $q : M_d(U) \to M_d(U)/\ker(T)$ is the quotient map. Then T' is also continuous, open and bijective. Hence,

$$T'^{-1} : \ell^\infty(E) \to M_d(U)/\ker(T),$$

is continuous and therefore bounded. Let $K > ||T'^{-1}||$. For each $\phi \in B(\ell^\infty(E))$, we let $T'^{-1}(\phi) = \mu + \ker(T) \in M_d(U)/\ker(T)$. We can find $\nu \in \ker(T)$ such that $||\mu + \nu||_{M(G)} \leq K$. Since $\nu \in \ker(T)$, $\widehat{\mu + \nu}$ and ϕ agree on E, proving (2).

(3) $\Rightarrow$ (1)

Suppose (3) holds. Let $\phi \in B(\ell^\infty(E))$ and $0 < \varepsilon < 1$. Choose $\mu_1 \in M_d(U)$ such that $||\mu_1|| \leq K$ and $||\phi - \widehat{\mu_1}||_\infty < \varepsilon$ on E. Note that the function $(\phi - \widehat{\mu_1})/\varepsilon \in B(\ell^\infty(E))$ and by (3) we can choose $\mu_2 \in M_d(U)$ with $||\mu_2|| < K$ such that $||(\phi - \widehat{\mu_1})/\varepsilon - \widehat{\mu_2}||_\infty < \varepsilon$ on E. Observe that $(\mu_1 + \varepsilon\mu_2) \in M_d(U)$ and $\left|\left|\phi - \widehat{\mu_1 + \varepsilon\mu_2}\right|\right|_\infty < \varepsilon^2$ on E. Iterating in this way, for each $n \in \mathbb{N}$ we can find $\mu_n \in M_d(U)$ such that $||\mu_n|| < K$ and $\left|\left|\phi - \widehat{\sum_{i=1}^n \varepsilon^{i-1}}\mu_i\right|\right|_\infty < \varepsilon^n$.

Let $\mu = \sum_{i=1}^\infty \varepsilon^{i-1}\mu_i$. As $||\mu|| \leq \sum_{i=1}^\infty \varepsilon^{i-1} ||\mu_i|| \leq \sum_{i=1}^\infty \varepsilon^{i-1} K < \infty$, $\mu \in M_d(U)$, and since $\phi = \widehat{\mu}$ on E, (1) is proven.

Remark. We call the iterative argument in the proof above the 'standard iteration'. This argument will be frequently used later.

The final step we need to show here is (3) implies (4). Before we prove (3) $\Rightarrow$ (4), we first prove (3) is equivalent to (3'):

(3') For each $0 < \varepsilon < 1$, there exists K such that for every (Hermitian) $\phi \in B(\ell^\infty(E))$, there exists $\mu \in M_d(U)$ ($\mu \in M_d^+(U)$) with $||\mu|| \leq K$ and $|\widehat{\mu}(\gamma) - \phi(\gamma)| < \varepsilon$ for all $\gamma \in E$.

Indeed, if (3) holds for some $0 < \varepsilon < 1$, then for each $0 < \varepsilon' < \varepsilon$, we let n be the integer that $\varepsilon^n < \varepsilon'$. If $\phi \in B(\ell^\infty(E))$, then after n steps of the iteration argument in the proof above, there is a discrete measure $\mu \in M_d(U)$ with $||\mu||_{M(G)} \leq \sum_{i=1}^n \varepsilon^{i-1}K$, such that $||\phi - \widehat{\mu}|_E||_\infty < \varepsilon^n < \varepsilon'$.

We thus give a proof for (3') $\Rightarrow$ (4) with any $\varepsilon < 1$ for the FZI_0 case.

Let $0 < \varepsilon < 1$ be fixed. For each $\gamma \in E$, define $\mathbb{D}_\gamma := [-1, 1]$ if $\gamma = \gamma^{-1}$, and $\mathbb{D}_\gamma := \overline{B_1}(\mathbb{C}) = \{z \in \mathbb{C} : |z| \leq 1\}$ if $\gamma \neq \gamma^{-1}$. Let $\mathbb{D}_E := \prod_{\gamma \in E} \mathbb{D}_\gamma$. For $n \in \mathbb{N}$, we define

$$D^+(n,U) := \{\mu \in M_d^+(U) \mid \mu = \sum_{i=1}^{n} a_i \delta_{x_i}, |a_i| \leq 1, x_i \in U\}$$

and

$$W_n := \{\phi \in \mathbb{D}_E \mid \exists \mu \in D^+(n,U) \text{ with } |\widehat{\mu}(\gamma) - \phi(\gamma)| \leq \varepsilon/2 \ \forall \gamma \in E\}.$$

From (3'), $\bigcup_n W_n = \mathbb{D}_E$.

Claim. Each W_n is closed in $\mathbb{D}_E$ with the product topology.

Fix $n \in \mathbb{N}$ and let $(\phi_\lambda)_{\lambda \in \Lambda}$ be a net in W_n such that $\phi_\lambda \to \phi \in \mathbb{D}_E$. For each $\lambda \in \Lambda$, we find $a_{\lambda,j} \in \overline{B_1}(\mathbb{C})$ (or $[-1,1]$ if $\gamma = \gamma^{-1}$) and $x_{\lambda,j} \in U$, $1 \leq j \leq n$ such that $||\phi_\lambda - \widehat{\mu_\lambda}|_E||_\infty \leq \varepsilon/2$, where $\mu_\lambda = \sum_{j=1}^n a_{\lambda,j}\delta_{x_{\lambda,j}}$. By passing to a subnet, if necessary, we may assume that $a_{\lambda,j} \to a_j \in \overline{B_1}(\mathbb{C})$ (or $[-1,1]$), and $x_{\lambda,j} \to x_j \in U$ for $1 \leq j \leq n$.

Let $\mu = \sum_{j=1}^n a_j \delta_{x_j}$. Since $x_{\lambda,j} \to x_j$ implies $\gamma(x_{\lambda,j}) \to \gamma(x_j)$ and $\phi_\lambda(\gamma) \to \phi(\gamma)$ for every γ, we have that, for each $\gamma \in E$ and for all $\varepsilon' > 0$,

$$\begin{aligned}
|\phi(\gamma) - \widehat{\mu}(\gamma)| &= |\phi(\gamma) - \sum_{j=1}^{n} a_j \overline{\gamma(x_j)}| \\
&\leq |\phi(\gamma) - \phi_\lambda(\gamma)| + |\phi_\lambda(\gamma) - \sum_{j=1}^{n} a_{\lambda,j}\overline{\gamma(x_{\lambda,j})}| \\
&\quad + |\sum_{j=1}^{n} a_{\lambda,j}\overline{\gamma(x_{\lambda,j})} - \sum_{j=1}^{n} a_j \overline{\gamma(x_j)}| \\
&< \varepsilon'/2 + \varepsilon/2 + \varepsilon'/2,
\end{aligned}$$

if λ is large enough. Since $\varepsilon' > 0$ is arbitrary, $|\phi(\gamma) - \widehat{\mu}(\gamma)| \leq \varepsilon/2$ and therefore $\phi \in W_n$, proving the Claim.

The Baire Category Theorem tells that some W_n will have non-empty interior.

Hence, there is some finite set $F \subset E$ and a point $(z_1, ..., z_{|F|})$ such that $(z_1, ..., z_{|F|}) \times \mathbb{D}_{E\setminus F} \subset W_n$.

Consider the real subspace S of $\ell^\infty(E)$ consisting of all the Hermitian functions which vanish off F. As F is finite, S is finite dimensional. Take a basis of S, say $e_1, ..., e_l$, where $e_j \in B(\ell^\infty(E))$, $1 \leq j \leq l$. Since all norms are equivalent on a finite dimensional space, there is some $c > 0$ that

$$\left|\left|\sum_{j=1}^{l} b_j e_j\right|\right|_{\ell^\infty} \geq c\sum_{j=1}^{l} |b_j|.$$

Each $\pm e_j$ is Hermitian, so again by (3') we can find $\mu_j, \nu_j \in M_d^+(U)$ with $|\widehat{\mu}_j(\gamma) - e_j(\gamma)| < c\varepsilon/4n$ and $|\widehat{\nu}_j - (-e_j)(\gamma)| < c\varepsilon/4n$ for $\gamma \in E$. Because we only have a finite number of μ_j's and ν_j's and each μ_j or ν_j is a norm limit of finite length, discrete measures, we can also assume that for some large enough m, $\mu_j, \nu_j \in D^+(m, U)$ for all j.

Let $\phi \in B(\ell^\infty(E))$ be a Hermitian function. Since ϕ coincides on $E\setminus F$ with some element in W_n, we can find $\mu \in D^+(n, U)$ such that $|\widehat{\mu}(\gamma) - \phi(\gamma)| \leq \varepsilon/2$ for all $\gamma \in E\setminus F$. As μ is a positive discrete measure, $(\phi - \widehat{\mu})|_F$ (extended by 0) belongs to S and therefore equals $\sum_{j=1}^{l} b_j e_j$ for some $b_j \in \mathbb{R}$. Write $b_j = b_j^+ - b_j^-$ where $b_j^\pm \geq 0$. Notice

$$c\sum_{j=1}^{l} |b_j| \leq ||\phi - \widehat{\mu}||_\infty \leq 1 + ||\mu|| \leq 2n.$$

For $\gamma \in E$,

$$\begin{aligned}
&|\phi(\gamma) - \widehat{\mu}(\gamma) - (\sum_{j=1}^{l}(b_j^+\widehat{\mu}_j + b_j^-\widehat{\nu}_j))(\gamma)| \\
=&|(\phi - \widehat{\mu})|_{E\setminus F}(\gamma) + (\phi - \widehat{\mu})|_F(\gamma) - (\sum_{j=1}^{l}(b_j^+\widehat{\mu}_j + b_j^-\widehat{\nu}_j))(\gamma)|
\end{aligned}$$

$$
\begin{aligned}
&=|(\phi-\widehat{\mu})|_{E\setminus F}(\gamma)+(\sum_{j=1}^{l}(b_j^+(e_j-\widehat{\mu}_j)+b_j^-(-e_j-\widehat{\nu}_j)))(\gamma)| \\
&\leq \sup_{\gamma\in E\setminus F}|\phi(\gamma)-\widehat{\mu}(\gamma)|+\sup_{\gamma\in E}|\sum_{j=1}^{l}(b_j^+(e_j-\widehat{\mu}_j)+b_j^-(-e_j-\widehat{\nu}_j)))(\gamma)| \\
&\leq \frac{\varepsilon}{2}+\frac{c\varepsilon}{4n}\sum_{j=1}^{l}|b_j|\leq\varepsilon.
\end{aligned}
$$

Note that $\mu+\sum_{j=1}^{l}b_j^+\mu_j+\sum_{j=1}^{l}b_j^-\nu_j\in D^+(N,U)$, where $N=n+2m\dim(F)$, which is independent of the choice of ϕ. $\square$

If $U\subset G$ is symmetric, it suffices to check that the ±1-valued functions on E can be approximated by Fourier transforms of discrete measures concentrated on U to prove E is $I_0(U)$. We have the following proposition.

Proposition 2.5 ([4])**.** *Let $U\subset G$ be Borel and symmetric and $E\subset\Gamma$. Suppose, for some $0<\varepsilon<1$, we have that for all $\varphi:E\to\{-1,1\}$ there is $\mu\in M_d(U)$ such that $|\varphi(\gamma)-\widehat{\mu}(\gamma)|<\varepsilon$ for all $\gamma\in E$. Then E is $I_0(U)$.*

Proof. First, let $\phi:E\to[-1,1]$ be given and define $\varphi:E\to\{-1,1\}$ via

$$
\varphi(\gamma)=\begin{cases}1 & \text{if } \phi(\gamma)\geq 0\\ -1 & \text{else}\end{cases}.
$$

Then $||\phi-\varphi/2||\leq 1/2$ on E.

By our assumption, there exists $\mu\in M_d(U)$ such that $||\widehat{\mu}-\varphi||<\varepsilon$ on E. Write $\mu=\sum_{i=1}^{\infty}a_i\delta_{x_i}$ where $x_i\in U$, and we let $\nu:=(\mu+\widetilde{\mu})/2$.

For each $\gamma\in E$,

$$
\begin{aligned}
\widehat{\nu}(\gamma)&=\frac{\widehat{\mu}(\gamma)+\widehat{\widetilde{\mu}}(\gamma)}{2}\\
&=\frac{\sum_{i=1}^{\infty}(a_i\overline{\gamma(x_i)}+\overline{a_i}\gamma(x_i))}{2}\in\mathbb{R}.
\end{aligned}
$$

Hence, ν has real Fourier transform. Moreover,

$$||\varphi - \widehat{\nu}|| \leq \frac{1}{2} ||\varphi - \widehat{\mu}|| + \frac{1}{2} \left|\left|\varphi - \widehat{\widetilde{\mu}}\right|\right| \leq \varepsilon.$$

Thus, $||\phi - \widehat{\nu}/2|| \leq 1/2 + \varepsilon/2 < 1$. Using the standard iteration, we have that there exists $\mu \in M_d(U)$ with $\widehat{\mu} = \phi$ on E.

For any $\phi \in B(\ell^\infty(E))$, we can interpolate the real and imaginary parts of ϕ in this way to see that E is $I_0(U)$. $\square$

Remark. Note that the proposition above immediately implies ε-Kronecker sets with $\varepsilon < 1$ are I_0.

In fact, any ε-Kronecker(U) set with $\varepsilon < \sqrt{2}$ is an $FZI_0(U)$ set.

Theorem 2.6 (ε-Kronecker sets are FZI_0, [5])**.** *Let $0 < \varepsilon < \sqrt{2}$ and $E \subset \Gamma$ be an ε-Kronecker(U) subset, where $U \subset G$ is Borel and symmetric. Then, E is $FZI_0(U)$.*

Remark. If E is ε-Kronecker for some $\varepsilon < \sqrt{2}$, then $E \cap E^{-1} = \emptyset$. So we do not need to worry about the Hermitian issue.

Notation: For $z \in \mathbb{C}$, $\Re(z)$ and $\Im(z)$ are the real and imaginary part of z respectively.

Proof. Choose a $0 < \delta < 1$, depending on ε, such that for $z \in \mathbb{T}$, $|z - 1| < \varepsilon$ implies $\Re(z) > \delta$. Also pick $b \in (0, \delta)$ such that $b + \sqrt{1 - \delta^2} < 1$.

Let $\varphi \in B(\ell^\infty(E))$. We write $\varphi(\gamma) = a_\gamma + ib_\gamma$ for each $\gamma \in E$, where $a_\gamma, b_\gamma \in [-1, 1]$. Find $x_0 \in U$ such that for all $\gamma \in E$,

$$\begin{aligned} |\gamma(x_0) - 1| < \varepsilon \quad & \text{if } a_\gamma \in [b, 1], \\ |\gamma(x_0) + 1| < \varepsilon \quad & \text{if } a_\gamma \in [-1, -b], \\ |\gamma(x_0) - i| < \varepsilon \quad & \text{if } a_\gamma \in (-b, b). \end{aligned}$$

If $a_\gamma \in [b, 1]$, then $|\gamma(x_0) - 1| < \varepsilon$ and therefore $\Re(\gamma(x_0)) > \delta$, which implies $|a_\gamma - \Re(\gamma(x_0))| < 1 - \delta$. If $a_\gamma \in [-1, -b]$, then $|\gamma(x_0) + 1| < \varepsilon$. This means

$\Re(\gamma(x_0)) < -\delta$ and hence $|a_\gamma - \Re(\gamma(x_0))| < 1-\delta$. Lastly, if $a_\gamma \in (-b, b)$, then $|\gamma(x_0) - i| < \varepsilon$, which gives $\Im(\gamma(x_0)) > \delta$. Hence, $|\Re(\gamma(x_0))| < \sqrt{1-\delta^2}$ and therefore $|a_\gamma - \Re(\gamma(x_0))| < b + \sqrt{1-\delta^2}$.

Thus, we have $|a_\gamma - \Re(\gamma(x_0))| < \max(1-\delta, b+\sqrt{1-\delta^2}) < 1$. By iterating we can interpolate any real sequence on E with the Fourier transform of some $\mu \in M_d^+(U)$.

To interpolate $\{ib_\gamma\}_{\gamma\in E}$, we argue as follows. Let $x_1 \in U$ be such that $|\gamma(x_1) - r_\gamma i| < \varepsilon$ for all $\gamma \in E$, where $r_\gamma = 1$ if $b_\gamma \geq 0$ and $r_\gamma = -1$ if $b_\gamma < 0$. Since $f(\gamma) := -\Re(\gamma(x_1)) \in B(\ell^\infty(E))$, from above there exists $\mu \in M_d^+(U)$ such that $\widehat{\mu} = f$ on E. Note that the $\delta > 0$ given above also satisfies $|\gamma(x) - i| < \varepsilon$ implies $\Im(\gamma(x)) > \delta$. Thus, if $b_\gamma \geq 0$, then $\Im(\gamma(x)) > \delta$ and if $b_\gamma < 0$, $\Im(\gamma(x)) < -\delta$. For each $\gamma \in E$, as $|\Im(\gamma(x_1))/2| \in (\delta/2, 1/2)$, $|b_\gamma| \in [0,1]$ and they have the same sign, we have

$$|\frac{\widehat{\mu + \delta_{x_1}}(\gamma)}{2} - ib_\gamma| = |\frac{\Im(\gamma(x_1))}{2} - b_\gamma| < 1 - \frac{\delta}{2}.$$

Hence, if $\sigma \in M_d^+(U)$ is the measure interpolating $\Re(\varphi)$, then $\tau := \sigma + (\mu+\delta_{x_1})/2 \in M_d^+(U)$ satisfies $|\widehat{\tau} - \varphi| < 1 - \delta/2 < 1$ on E. We get the desired measure by the standard iteration argument. $\square$

We now give examples of ε-Kronecker sets.

Definition 2.7. A set $E = \{x_i\}_{i=1}^\infty \subset \mathbb{R}^+$ is Hadamard with ratio $q > 1$ if for each $i \geq 1$ we have $x_{i+1}/x_i \geq q$.

Hadamard sets with ratio $q > 2$ are examples of ε-Kronecker sets for $\varepsilon > |1 - e^{i\pi/(q-1)}|$, as the following proposition implies.

Proposition 2.8. *Suppose $E = (n_j)_{j=1}^\infty \subset \mathbb{Z}^+$ is Hadamard with ratio $q > 2$. For each $\varphi : E \to \mathbb{T}$ there exists θ such that for all j,*

$$|\varphi(n_j) - e^{in_j\theta}| \leq |1 - e^{i\pi/(q-1)}|.$$

Proof. Let $\varphi : E \to \mathbb{T}$. Since e^{in_jx} takes all values in $\mathbb{T}$ as x ranges over an interval (in $\mathbb{T}$) of length at least $2\pi/n_j$, we can choose θ_j inductively by the following procedure. We choose $\theta_1 \in [-\pi/n_1, \pi/n_1]$ such that $\varphi(n_1) = e^{in_1\theta_1}$. For $j \geq 2$, we choose $\theta_j \in [\theta_{j-1} - \pi/n_j, \theta_{j-1} + \pi/n_j]$ such that $\varphi(n_j) = e^{in_j\theta_j}$.

By this construction, we have that for $1 \leq j < k$,

$$\begin{aligned} |\theta_j - \theta_k| &\leq \sum_{l=j+1}^{k} |\theta_{l-1} - \theta_l| \leq \sum_{l=j+1}^{k} \frac{\pi}{n_l} \\ &\leq \sum_{l=1}^{\infty} \frac{\pi}{n_j q^l} = \frac{\pi}{n_j(q-1)}. \end{aligned}$$

Since $n_j \to \infty$, this means $(\theta_j)_{j=1}^{\infty}$ is Cauchy. Let $\theta_j \to \theta$. For each j, $|\theta - \theta_j| \leq |\theta - \theta_k| + |\theta_k - \theta_j|$ for all $k > j$, so $|\theta - \theta_j| \leq \pi/(n_j(q-1))$. Therefore,

$$\begin{aligned} |\varphi(n_j) - e^{in_j\theta}| &\leq |\varphi(n_j) - e^{in_j\theta_j}| + |e^{in_j\theta_j} - e^{in_j\theta}| \\ &\leq |1 - e^{i\pi/(q-1)}|, \end{aligned}$$

as desired. $\square$

Remark. Theorem 2.8 implies Hadamard sets with ratio greater than three are FZI_0. It is known that every Hadamard set is I_0 [14].

We next provide a property of ε-Kronecker sets for later use.

Proposition 2.9. *Let $E \subset \Gamma$ be a subset. Suppose each finite subset $F \subset E$ is ε-Kronecker. Then, E is weak ε-Kronecker.*

Proof. Let $\varphi : E \to \mathbb{T}$ be a function. We may assume E is infinite. For each $F \subset E$ finite, we find $x_F \in G$ such that $|\gamma(x_F) - \varphi(\gamma)| < \varepsilon$ for all $\gamma \in F$. We partially order the collection $\mathcal{F} := \{F \subset E : |F| < \infty\}$ by inclusion and note that $(x_F)_{|F|<\infty}$ forms a net in G. Since G is compact, we let x be a cluster point of $(x_F)_{|F|<\infty}$. For any fixed $\gamma \in E$ and for any $\delta > 0$, we may choose F' large enough that $\gamma \in F'$ and $|\gamma(x) - \gamma(x_{F'})| < \delta$. Then,

$$|\gamma(x) - \varphi(\gamma)| \leq |\gamma(x) - \gamma(x_{F'})| + |\gamma(x_{F'}) - \varphi(\gamma)| < \delta + \varepsilon.$$

As $\delta > 0$ is arbitrary, this means E is weak ε-Kronecker. $\square$

We also notice that the interpolation property of a subset $E \subset \Gamma$ is invariant under translations.

Proposition 2.10. *Let $U \subset G$ be a Borel set and $E \subset \Gamma$. If E is $I_0(U)$, then*
(1) $E\gamma$ is $I_0(U)$ for every $\gamma \in \Gamma$, and
(2) E is $I_0(Ux)$ for every $x \in G$.

Proof. (1) Let $\phi : E\gamma \to \mathbb{C}$ be bounded. Define $\varphi : E \to \mathbb{C}$ via $\varphi(\beta) = \phi(\gamma \cdot \beta)$. Since E is $I_0(U)$, there exists a discrete measure $\mu = \sum_i a_i \delta_{x_i}$, where $x_i \in U$, such that $\widehat{\mu} = \varphi$ on E. Let $\nu := \sum_i a_i \gamma(x_i) \delta_{x_i}$. Then, for every $\beta \in E$, we have $\widehat{\nu}(\beta \cdot \gamma) = \sum_i a_i \gamma(x_i) \overline{\beta(x_i)\gamma(x_i)} = \sum_i a_i \overline{\beta(x_i)} = \widehat{\mu}(\beta) = \varphi(\beta) = \phi(\beta \cdot \gamma)$. Hence, $E\gamma$ is $I_0(U)$, too.

(2) Let $\phi : E \to \mathbb{C}$ be bounded. Note that the function $\varphi(\gamma) = \phi(\gamma)\gamma(x)$ is also a bounded function from E to $\mathbb{C}$ and therefore there exists a discrete measure $\mu = \sum_i a_i \delta_{x_i}$ with $x_i \in U$ and $\widehat{\mu} = \varphi$. Then, $\nu = \sum_i a_i \delta_{x_i x}$ will be a discrete measure concentrated on Ux and $\widehat{\nu} = \phi$. $\square$

For ε-Kronecker sets, however, we only have the following.

Proposition 2.11. *If $E \subset \Gamma$ is ε-Kronecker(U), then E is ε-Kronecker(Ux) for every $x \in G$.*

Proof. If $E \subset \Gamma$ is ε-Kronecker(U), then for any function $\phi : E \to \mathbb{T}$, $\phi/\widehat{x}$, where $\widehat{x}(\gamma) = \gamma(x)$, is still a function from E to $\mathbb{T}$. Since E is ε-Kronecker(U), there exists $x_0 \in U$ such that for every $\gamma \in E$, $|\phi(\gamma)/\gamma(x) - \gamma(x_0)| < \varepsilon$. Hence, $|\phi(\gamma) - \gamma(x_0 x)| < \varepsilon$ and $x_0 x \in Ux$. This means E is also ε-Kronecker(Ux). $\square$

Remark. Notice that the analogue of (1) in Proposition 2.10 fails for FZI_0 sets since the singleton $\{0\} \subset \mathbb{Z}$ is not FZI_0, while all other singletons in $\mathbb{Z}$ are.

2.2 I_0 sets and the Bohr topology

Another characterization of I_0 sets is related to the Bohr compactification of Γ, due to Hartman and Ryll-Nardzewski.

Theorem 2.12 (HRN characterization, [11]). *The following are equivalent.*
(1) E is I_0.
(2) If E_1 and E_2 are disjoint in E, then they have disjoint closures in the Bohr compactification of Γ.
(3) If E_1 and E_2 are disjoint in E, then there is a discrete measure μ such that $\widehat{\mu}$ is 1 on E_1 and is 0 on E_2.

Proof. (1) $\Rightarrow$ (2)
Let $\phi : E \to \mathbb{C}$ be such that ϕ is 0 on E_1 and is 1 on E_2. Find a discrete measure μ with $\widehat{\mu} = \phi$ on E. Since $\widehat{\mu}$ is continuous on the Bohr compactification of Γ, we see that E_1 and E_2 must have disjoint closures.

(2) $\Rightarrow$ (3)
Suppose E_1 and E_2 are disjoint sets in E and their Bohr closures are disjoint. Since $\overline{\Gamma}$ is both Hausdorff and compact, $\overline{\Gamma}$ is normal.

Claim. There exists an open set $V \subset \overline{\Gamma}$ such that $\overline{E_1} \cdot V \cap \overline{E_2} \cdot V = \emptyset$.

From the normality of $\overline{\Gamma}$, since $\overline{E_1} \cap \overline{E_2} = \emptyset$, there exists U_1, U_2 open in $\overline{\Gamma}$ such that $U_1 \cap U_2 = \emptyset$ with $\overline{E_1} \subset U_1$ and $\overline{E_2} \subset U_2$. Since $\overline{\Gamma}$ is compact and Hausdorff and $\overline{E_1} \subset \overline{\Gamma}$ is closed, $\overline{E_1}$ is also compact.
Note that $\overline{E_1} \subset U_1$ implies $1 \notin \overline{E_1}^{-1} \cdot U_1^c$ and $\overline{E_1}^{-1} \cdot U_1^c$ is compact. By normality, find an open set V_1 containing 1 such that $V_1 \cap \overline{E_1}^{-1} \cdot U_1^c = \emptyset$. This implies $\overline{E_1} \cdot V_1 \subset U_1$. Similarly, we can find open V_2 such that $\overline{E_2} \cdot V_2 \subset U_2$. Finally, we take $V = V_1 \cap V_2$ and the Claim is proved.

Now we return to proving (2) $\Rightarrow$ (3). Take $g, h \in \ell^2(G_d)$ such that $\widehat{g} = 1_V$ and $\widehat{h} = 1_{\overline{E_1} \cdot V^{-1}}$. Let $d\mu = g \cdot h / m_{\overline{\Gamma}}(V) \in \ell^1(G_d) = M_d(G)$. Since $\widehat{\mu} = (\widehat{g} * \widehat{h})/m_{\overline{\Gamma}}(V)$, $\widehat{\mu}$ is 1 on $\overline{E_1}$ and is 0 off $\overline{E_1} \cdot V \cdot V^{-1}$, and in particular, $\widehat{\mu}$ is 0 on $\overline{E_2}$.

(3) $\Rightarrow$ (1)
Suppose that for any disjoint sets E_1 and E_2 in E, we can find discrete μ such that $\widehat{\mu}$ is 1 on E_1 and 0 on E_2. Let $\phi : E \to \{-1, 1\}$. Then $E_1 := \phi^{-1}(\{1\})$ and $E_2 := \phi^{-1}(\{-1\})$ are disjoint and there is a discrete μ_1 with $\widehat{\mu_1} = 1$ on E_1 and $\widehat{\mu_1} = 0$ on E_2. Also, there is another discrete measure μ_2 with $\widehat{\mu_2} = 0$ on E_1

and $\widehat{\mu_2} = -1$ on E_2. We have that $\phi = \widehat{\mu_1} + \widehat{\mu_2} = \widehat{\mu_1 + \mu_2}$ on E. Since we can interpolate ± 1 functions on E, from Proposition 2.5 we have that E is I_0. $\square$

The interpolation property of an I_0 set E will continue to hold if we add a finite number of elements to E. The proof of this uses the fact that any I_0 set does not cluster in $\overline{\Gamma}$ at any continuous character. Towards proving that, we need the following lemma.

Lemma 2.13. *Let $E \subset \Gamma$ be an I_0 set. There exist $N \geq 1$, a finite set $F \subset E$ and a fixed set $\{c_1, ..., c_N\} \subset \Delta := \{z \in \mathbb{C} \mid |z| \leq 1\}$ such that for all $\varphi : E \backslash F \to \{-1, 1\}$, there are $\{x_1, ..., x_N\} \subset G$ with $|\varphi(\gamma) - \sum_{i=1}^{N} c_i \gamma(x_i)| \leq 1/3$.*

Proof. For each $M \geq 1$, let D_M be a countable dense subset of Δ^M. For each $M \geq 1$ and $c = (c_1, ..., c_M) \in D_M$, let $\widetilde{\mathrm{AP}}(M, c)$ be the set of all $\varphi \in B(\ell^\infty(E)) = \Delta^E$ such that there exist $x_1, ..., x_M \in G$ with $|\varphi(\gamma) - \sum_{i=1}^{M} c_i \gamma(x_i)| \leq 1/3$ for all $\gamma \in E$. Since G is compact, each $\widetilde{\mathrm{AP}}(M, c)$ is closed in Δ^E.

Moreover, since E is I_0, for each $\phi \in B(\ell^\infty(E))$, there exists $\mu = \sum_{i=1}^{\infty} a_i \delta_{x_i} \in M_d(G)$ such that $\widehat{\mu} = \phi$, where $a_i \in \Delta$. Hence, there exists $M_0 \in \mathbb{N}$ such that $|\sum_{i=1}^{M_0} a_i \gamma(x_i^{-1}) - \phi(\gamma)| < 1/3$. Since D_{M_0} is dense in Δ^{M_0}, we may take $(c_1, ..., c_{M_0}) \in D_{M_0}$ such that $|\sum_{i=1}^{M_0} c_i \gamma(x_i^{-1}) - \phi(\gamma)| \leq 1/3$. This means $\bigcup_{M \geq 1} \bigcup_{c \in D_M} \widetilde{\mathrm{AP}}(M, c) = B(\ell^\infty(E))$.

The Baire Category theorem implies there is some $N \geq 1$ and $c \in D_N$ with $\widetilde{\mathrm{AP}}(N, c)$ having non-empty interior. By the definition of the product topology, there exists a finite set $F \subset E$ such that $\widetilde{\mathrm{AP}}(N, c)$ contains $V \times \Delta^{E \backslash F}$, where $V \subset \Delta^F$ is open. This means for any $\varphi : E \backslash F \to \{-1, 1\}$, $|\varphi(\gamma) - \sum_{i=1}^{N} c_i \gamma(x_i)| \leq 1/3$ for all $\gamma \in E \backslash F$. $\square$

Theorem 2.14 ([16])**.** *If E is an I_0 set, then E does not cluster in the Bohr topology at any continuous characters.*

Proof. It is enough to show E does not cluster at 1. Suppose, otherwise, that E clusters at 1. By Lemma 2.13, there exists a finite set $F \subset E$ and $c_1, ..., c_N \in \Delta$

such that $|1 - \sum_{i=1}^{N} c_i\gamma(x_i)| \leq 1/3$ for some $x_1, ..., x_N \in G$ and for all $\gamma \in E\backslash F$. Also, $|1 + \sum_{i=1}^{N} c_i\gamma(y_i)| \leq 1/3$ for some $y_1, ..., y_N \in G$ and for all $\gamma \in E\backslash F$.

Since E clusters at 1, $E\backslash F$ still clusters at 1. Since $\widehat{\sum_{i=1}^{N} c_i\delta_{x_i^{-1}}}$ and $\widehat{\sum_{i=1}^{N} c_i\delta_{y_i^{-1}}}$ are continuous functions on $\overline{\Gamma}$, there exists $\beta \in E\backslash F$ such that $|\sum_{i=1}^{N} c_i 1(x_i) - \sum_{i=1}^{N} c_i\beta(x_i)| < 1/7$ and $|\sum_{i=1}^{N} c_i 1(y_i) - \sum_{i=1}^{N} c_i\beta(y_i)| < 1/7$. Thus,

$$\begin{aligned} |1 - \sum_{i=1}^{N} c_i| &\leq |1 - \sum_{i=1}^{N} c_i\beta(x_i)| + |\sum_{i=1}^{N} c_i 1(x_i) - \sum_{i=1}^{N} c_i\beta(x_i)| \\ &< 1/3 + 1/7 < 1/2. \end{aligned}$$

Similarly, we have $|1 + \sum_{i=1}^{N} c_i| < 1/2$ at the same time, which is not possible. □

Remark. It is worth noting that a finite union of I_0 sets may not be I_0 [15]. Consider the disjoint sets $E_1 = \{10^j : j \in \mathbb{N}\}$ and $E_2 = \{10^j + j : j \in \mathbb{N}\}$. Both E_1 and E_2 are I_0 since they are Hadamard with ratio greater than three. Since $\mathbb{N}$ is dense in $\overline{\mathbb{Z}}$, we let $j_\alpha \to 0$ in $\overline{\mathbb{Z}}$ for $j_\alpha \in \mathbb{N}$. Since $\overline{\mathbb{Z}}$ is compact, by passing to a subnet if necessary, we assume $10^{j_\alpha} \to \chi \in \overline{\mathbb{Z}}$. Hence, both E_1 and E_2 cluster at χ in $\overline{\mathbb{Z}}$, which implies $E_1 \cup E_2$ is not I_0 by Theorem 2.12.

Proposition 2.15. *If E is I_0 and $F \subset \Gamma$ is a finite set, then $E \cup F$ is still I_0.*

Proof. Since F is finite, $\overline{F} = F$, where $\overline{F}$ is the closure of F in the Bohr topology on $\overline{\Gamma}$. Since $(\overline{E}\backslash E) \cap \overline{F} = (\overline{E}\backslash E) \cap F = \emptyset$ from the theorem above, the result holds from the HRN characterization of I_0 sets. □

If G is further assumed to be connected and $F \subset \Gamma$ finite, then any $E \in I_0(U)$ has the property that $E \cup F \in I_0(U \cdot W)$ for any W a neighbourhood of $e \in G$. That is, to interpolate an $I_0(U)$ set with a finite number of extra elements, we do not have to enlarge U too much.

Theorem 2.16 ([4]). *Let G be a connected compact abelian group and $U \subset G$ be a neighbourhood. If $E \subset \Gamma$ is $I_0(U)$, $\lambda \in \Gamma$ and $W \subset G$ is any neighbourhood of the identity, then $E \cup \{\lambda\}$ is $I_0(U \cdot W)$.*

Proof. By Proposition 2.10, we may assume $\lambda = 1 \in \Gamma$ and U is a neighbourhood of $e \in G$.

Claim. There exists a finitely supported measure $\mu = \sum_{j=1}^{J} c_j \delta_{x_j} \in M_d(G)$, $c_j \in \mathbb{C}$, $x_j \in G$, $J \in \mathbb{N}$, such that $\widehat{\mu}(1) = 1$ and $\widehat{\mu}(\gamma) < 1/100$ for all $\gamma \in E$.

From Proposition 2.15, since E is I_0, $E \cup \{1\}$ is also I_0 and therefore there exists $\mu' \in M_d(G)$ with $\widehat{\mu'}(\gamma) = 0$ for all $\gamma \in E$ and $\widehat{\mu'}(1) = 1$. Write $\mu' = \sum_{j=1}^{\infty} a_j \delta_{x_j}$ with $a_j \in \mathbb{C}$ and $x_j \in G$. Since $\{a_j\}_{j=1}^{\infty} \in \ell^1$, find $J \in \mathbb{N}$ such that $\sum_{j=J+1}^{\infty} |a_j| < 1/1000$. Put $\beta = \sum_{j=1}^{J} a_j \in \mathbb{C}$.

Let $\mu = (\sum_{j=1}^{J} a_j \delta_{x_j})/\beta$. Then, $\widehat{\mu}(1) = (\sum_{j=1}^{J} a_j)/\beta = 1$. Moreover, we notice that for all $\gamma \in E$,

$$|\widehat{\sum_{j=1}^{J} a_j \delta_{x_j}}(\gamma)| = |\widehat{\mu'}(\gamma) - \widehat{\sum_{j=J+1}^{\infty} a_j \delta_{x_j}}(\gamma)| \leq |\widehat{\mu'}(\gamma)| + |\widehat{\sum_{j=J+1}^{\infty} a_j \delta_{x_j}}(\gamma)|$$
$$= |\widehat{\sum_{j=J+1}^{\infty} a_j \delta_{x_j}}(\gamma)| \leq \sum_{j=J+1}^{\infty} |a_j| < 1/1000.$$

Similarly, $|\beta| \geq 999/1000$. Hence, $|\widehat{\mu}(\gamma)| = |\widehat{\sum_{j=1}^{J} a_j \delta_{x_j}}(\gamma)|/|\beta| \leq 1/999 < 1/100$, proving the Claim.

Let V be an open neighbourhood of $e \in G$ such that $V \cdot V^{-1} \subset W$.

Claim. $G = \bigcup_{n=1}^{\infty} V^n$.

Let $U_0 \subset V$ be a symmetric open neighbourhood of $e \in G$. Denote $W := \bigcup_{n=1}^{\infty} U_0^n$. Notice that since each U_0^n is open, W is open. Moreover, if $x, y \in W$, write $x = \prod_{i=1}^{n} u_i$ and $y = \prod_{j=1}^{m} v_j$ for $u_i, v_j \in U_0$. Then, $xy = \prod_{i,j} u_i v_j \in U_0^{n+m} \subset W$. Also, $x^{-1} = \prod_{i=1}^{n} u_i^{-1} \in U_0^n \subset W$, as U_0 is symmetric. This means W is an open subgroup and therefore is closed. But G is connected and clearly $W \neq \emptyset$ ($e \in W$), so $W = G$. Hence, $W \subset \bigcup_{n=1}^{\infty} V^n$ implies $\bigcup_{n=1}^{\infty} V^n = G$, proving the Claim.

Therefore there exists an integer N such that $x_j \in V^N$ for all $j = 1, ..., J$. Write $x_j = \prod_{k=1}^N w_{j,k}$ with $w_{j,k} \in V$.

Claim. For some $\varepsilon > 0$,

$$\sum_{j=1}^{J} \sum_{k=1}^{N} |\gamma(w_{j,k}) - 1(w_{j,k})| \geq \varepsilon$$

for all $\gamma \in E$.

We first justify the following inequality. If $a, b \in \mathbb{T}$, then

$$|ab - 1| \leq |a - 1| + |b - 1|. \tag{$*$}$$

Indeed, notice that $|ab-1| = |ab-a+a-1| \leq |ab-a|+|a-1| = |a||b-1|+|a-1| = |a-1| + |b-1|$.

As $|\widehat{\mu(\gamma)}| < 1/100$ for all $\gamma \in E$, we have

$$\begin{aligned} 99/100 < |\widehat{\mu}(\gamma) - 1| = |\widehat{\mu}(\gamma) - \widehat{\mu}(1)| &= |\sum_{j=1}^{J} c_j(\overline{\gamma(x_j)} - 1)| \\ &\leq \sum_{j=1}^{J} |c_j||\overline{\gamma(x_j)} - 1| = \sum_{j=1}^{J} |c_j||\prod_{k=1}^{N} \overline{\gamma(w_{j,k})} - 1|. \end{aligned} \tag{$**$}$$

Denote $C = \max |c_j|$. Using (*), we have

$$\begin{aligned} (**) \leq C \sum_{j=1}^{J} \sum_{k=1}^{N} |\overline{\gamma(w_{j,k})} - 1| &= C \sum_{j=1}^{J} \sum_{k=1}^{N} |\gamma(w_{j,k}) - 1| \\ &= C \sum_{j=1}^{J} \sum_{k=1}^{N} |\gamma(w_{j,k}) - 1(w_{j,k})|. \end{aligned}$$

Therefore, $\sum_{j=1}^J \sum_{k=1}^N |\gamma(w_{j,k}) - 1(w_{j,k})| \geq 99/(100C)$ for all $\gamma \in E$, proving the Claim with $\varepsilon = 99/(100C)$.

Thus, for all $\gamma \in E$, the Cauchy-Schwarz inequality implies

$$\begin{aligned}
& JN \cdot \left[\sum_{j=1}^{J} \sum_{k=1}^{N} (\delta_{w_{j,k}} - \delta_1) * (\delta_{w_{j,k}^{-1}} - \delta_1) \right]^{\widehat{}} (\gamma) \\
&= JN \cdot \left(\sum_{j=1}^{J} \sum_{k=1}^{N} (\overline{\gamma(w_{j,k})} - 1)(\gamma(w_{j,k}) - 1) \right) \\
&= JN \cdot \left(\sum_{j=1}^{J} \sum_{k=1}^{N} |\gamma(w_{j,k}) - 1|^2 \right) \\
&= \left(\sum_{j=1}^{J} \sum_{k=1}^{N} 1^2 \right) \cdot \left(\sum_{j=1}^{J} \sum_{k=1}^{N} |\gamma(w_{j,k}) - \gamma(w_{j,k})|^2 \right) \\
&\geq \left(\sum_{j=1}^{J} \sum_{k=1}^{N} |\gamma(w_{j,k}) - 1(w_{j,k})| \right)^2 \geq \varepsilon^2,
\end{aligned}$$

and hence

$$\left[\sum_{j=1}^{J} \sum_{k=1}^{N} (\delta_{w_{j,k}} - \delta_1) * (\delta_{w_{j,k}^{-1}} - \delta_1) \right]^{\widehat{}} (\gamma) \geq \frac{\varepsilon^2}{JN}.$$

Let $\omega = \sum_{j=1}^{J} \sum_{k=1}^{N} (\delta_{w_{j,k}} - \delta_1) * (\delta_{w_{j,k}^{-1}} - \delta_1)$. Then, $\widehat{\omega}$ is at least $\varepsilon^2/(JN)$ on E from above. Also, $\widehat{\omega}(1) = \sum_{j=1}^{J} \sum_{k=1}^{N} (\overline{1(w_{j,k})} - 1)(1(w_{j,k}) - 1) = 0$. Moreover, ω is supported on $V \cdot V^{-1} \subset W$.

Since E is $I_0(U)$ and $1/\widehat{\omega}$ is bounded on E (by JN/ε^2), there is $\nu \in M_d(U)$ such that $\widehat{\nu} = 1/\widehat{\omega}$ on E. Therefore, $\tau = \omega * \nu \in M_d(U \cdot W)$ and $\widehat{\tau}$ is 1 on E, 0 at 1.

To interpolate any $\phi \in B(\ell^\infty(E \cup \{1\}))$, we first find $\lambda \in M_d(U)$ with $\widehat{\lambda} = \phi$ on E, which we can do since E is $I_0(U)$. Let $\sigma = \lambda + (\phi(1) - \widehat{\lambda}(1))(\delta_e - \tau) \in M_d(U \cdot W)$. We have $\widehat{\sigma}(\gamma) = \widehat{\lambda}(\gamma) = \phi(\gamma)$ for all $\gamma \in E$ and $\widehat{\sigma}(1) = \widehat{\lambda}(1) + \phi(1) - \widehat{\lambda}(1) = \phi(1)$. This implies $E \cup \{1\}$ is $I_0(U \cdot W)$. □

The following results come directly from the theorem above.

Corollary 2.17. *Let G be a connected, compact abelian group and $U \subset G$ be* *a neighbourhood. If $E \subset \Gamma$ is $I_0(U)$, F is a finite set in Γ and $W \subset G$ any neighbourhood of the identity, then $E \cup F$ is $I_0(U \cdot W)$.*

Proof. Let W be a neighbourhood of $e \in G$. Let W_0 be another neighbourhood around e such that $W_0^{|F|} \subset W$. From Theorem 2.16, for any $\gamma \in F$, $E \cup \{\gamma\}$ is $I_0(U \cdot W_0)$. By repeated application, $E \cup F$ is $I_0(U \cdot W_0^{|F|})$ and therefore is $I_0(U \cdot W)$. $\square$

Corollary 2.18. *Let G be a connected, compact abelian group. Let E be a finite set in Γ. Then E is $I_0(U)$ for all open sets U.*

Proof. The empty set $\emptyset \subset \Gamma$ is $I_0(U)$ for all open $U \subset G$ and therefore this corollary follows immediately from the corollary above. $\square$

Theorem 2.19 ([4])**.** *Let $0 < \varepsilon' < \varepsilon$ and let $E \subset \Gamma$ be an ε'-Kronecker subset. For any compact neighbourhood $U \subset G$, there exists a finite set F so that $E \backslash F$ is ε-Kronecker(U).*

Proof. Since G is compact and U is a neighbourhood, there exists a finite collection of $\{x_n\}_{n=1}^N \subset G$ such that $\bigcup_{n=1}^N (x_n U) = G$. Because U is compact, each

$$X_n := \{\phi \in \mathbb{T}^E \mid \exists x \in x_n U \text{ with } |\phi(\gamma) - \gamma(x)| \leq \varepsilon' \ \forall \gamma \in E\}$$

is closed in $\mathbb{T}^E$ with the product topology. As E is ε'-Kronecker, $\bigcup_{n=1}^N X_n = \mathbb{T}^E$. This implies at least one of the X_n's has interior, say X_k. Thus, there exists a finite set $F \subset E$ such that $X_k \supset V \times \mathbb{T}^{E \backslash F}$, where $V \subset \mathbb{T}^F$ is open. Hence, for any $\phi \in \mathbb{T}^{E \backslash F}$, there exists $x \in x_k U$ such that $|\phi(\gamma) - \gamma(x)| \leq \varepsilon' < \varepsilon \ \forall \gamma \in E \backslash F$. Thus, $E \backslash F$ is ε-Kronecker$(x_k U)$ and therefore is ε-Kronecker(U) by Proposition 2.11. $\square$

We finish this chapter with some consequences of the previous theorem.

Corollary 2.20. *Suppose $E \subset \Gamma$ is weak ε-Kronecker, $\varepsilon' > \varepsilon$ and $\gamma \in \Gamma$. There exists a finite set $F \subset E$ such that $\gamma(E \backslash F)$ is ε'-Kronecker.*

Proof. Let $U = \{x \in G : |\gamma(x) - 1| \leq \varepsilon' - \varepsilon\}$. As U is a compact neighbourhood, by Theorem 2.19 there exists a finite set $F \subset E$ such that $E \backslash F$ is weak

ε-Kronecker(U).

Let $\varphi : \gamma(E\backslash F) \to \mathbb{T}$ be a function. We define $\phi : E\backslash F \to \mathbb{T}$ via $\phi(\beta) = \varphi(\gamma\beta)$. There exists $x \in U$ such that $|\phi(\beta) - \beta(x)| \le \varepsilon$ for all $\beta \in E\backslash F$. Furthermore, for all $\beta \in E\backslash F$, we have

$$|\varphi(\gamma\beta) - \gamma(x)\beta(x)| \le |\varphi(\gamma\beta) - \beta(x)| + |\beta(x) - \gamma(x)\beta(x)| < \varepsilon + \varepsilon' - \varepsilon = \varepsilon'.$$

$\square$

Corollary 2.21. *If $E \subset \Gamma$ is ε-Kronecker for $\varepsilon < \sqrt{2}$, then for each compact neighbourhood $U \subset G$, there exists a finite set $F \subset E$ such that $E\backslash F$ is $FZI_0(U)$.*

Proof. Let $U \subset G$ be a compact subset. From Theorem 2.19, there exists a finite set $F \subset E$ such that $E\backslash F$ is ε'-Kronecker(U) for some $\varepsilon' < \sqrt{2}$ and therefore $E\backslash F$ is $FZI_0(U)$ by Theorem 2.6. $\square$

Remark. In the next chapter, we will see that most infinite sets contain infinite ε-Kronecker sets for $\varepsilon < \sqrt{2}$ and hence contain infinite $FZI_0(U)$ sets for any compact neighbourhood $U \subset G$.

Chapter 3

Existence of interpolation sets

In this chapter, we will show that most infinite sets contain subsets of the same cardinality that are $FZI_0(U)$ for any given compact neighbourhood $U \subset G$.

3.1 Statement of the existence result

We first give a technical definition.

Notation: For a subset $F \subset \Gamma$, we denote by $\langle F \rangle$ the subgroup generated by F. We define $q_F : \Gamma \to \Gamma/\langle F \rangle$ to be the quotient map. For $2 \leq N \in \mathbb{N}$, we define Γ_N to be the subgroup of Γ whose elements have orders dividing N. We write $q_N = q_{\Gamma_N}$. Finally, we define $\Gamma_0 \subset \Gamma$ to be the subgroup of finite order elements and $q_0 : \Gamma \to \Gamma/\Gamma_0$ to be the quotient map.

Definition 3.1. Let $2 \leq N \in \mathbb{N}$ and $E \subset \Gamma$. We say that E is N-large if $|q_N(E)| < |E|$. If E is not N-large, we say that E is N-small. We say that E is tor-large if $|q_0(E)| < |E|$.

We state the main theorem of this chapter.

Theorem 3.2 ([7]). *Let $E \subset \Gamma$ be an infinite subset.*
(1) If E is N-small for all $N \geq 2$, then for all $\varepsilon > 0$ there exists an ε-Kronecker subset $F \subset E$ with $|F| = |E|$.
(2) Suppose E is M-large for some M. Let N be the smallest such M, L be any

prime number that divides N and k be the largest power such that $L^k|N$. Then E contains a weak ε-Kronecker subset F with $|F| = |E|$ and $\varepsilon = |1 - e^{i\pi/(L^k)}|$.

Corollary 3.3. *Let $E \subset \Gamma$ be an infinite subset. Suppose E is not 2-large.*
(1) For any compact neighbourhood $U \subset G$ there exists $F \subset E$ with $|F| = |E|$ such that F is $FZI_0(U)$.
(2) If G is connected, then there exists $F \subset E$ with $|F| = |E|$ such that F is $I_0(V)$ for all compact neighbourhoods $V \subset G$.

Proof. Since E is not 2-large, either E is N-small for all $N \geq 1$ or E is M-large for some minimal $M > 2$. In either case, Theorem 3.2 implies there exists $F_0 \subset E$ with $|F_0| = |E|$ such that F_0 is ε-Kronecker for $\varepsilon < \sqrt{2}$ (even weak ε-Kronecker for $\varepsilon \leq 1$). This F_0 will be used for the proofs of both (1) and (2).

(1) By Proposition 2.11, we may assume U is a compact neighbourhood of the identity in G. Since every neighbourhood of the identity contains a symmetric neighbourhood, we may further assume U is also symmetric.

Theorem 2.19 implies there exists a finite set $S \subset F_0$ such that $F_0 \backslash S$ is ε-Kronecker(U). Since $\varepsilon < \sqrt{2}$, by Theorem 2.6 $F_0 \backslash S$ is $FZI_0(U)$. Since S is finite, we have $|F_0 \backslash S| = |F_0| = |E|$.

(2) By Proposition 2.10, we may assume V is a compact neighbourhood around $e \in G$. Let $W \subset G$ be another compact neighbourhood around e such that $W^2 \subset V$. By Theorem 2.19, there exists a finite subset $C \subset F_0$ such that $F_0 \backslash C$ is ε'-Kronecker(W) for $\varepsilon' < \sqrt{2}$. By Theorem 2.6, $F_0 \backslash C$ is $I_0(V)$. By Corollary 2.17, $F_0 = (F_0 \backslash C) \cup C$ is $I_0(W^2)$ and therefore is $I_0(V)$ with $|F_0| = |E|$. $\square$

Remark ([7]). When E is infinite and 2-large, for any compact neighbourhood $U \subset G$ there exists $F \subset E$ with $|F| = |E|$ such that every ± 1-valued function on F can be interpolated exactly on U. By Proposition 2.5, F is $I_0(U)$.

To prove this theorem, we first establish some intermediate results.

3.2 Preliminaries to the proof of the existence theorem

The proof of Theorem 3.2 is based upon ideas of [5] and [7].

We first give a lemma that will allow us work in subgroups or enlarge to a larger group.

Lemma 3.4. *Let $E \subset \Gamma$, $\varepsilon > 0$, $\gamma \in \Gamma$ and $\Lambda \subset \Gamma$ be a subgroup.*
(1) Let $q : \Gamma \to \Gamma/\Lambda$ be the quotient map. If q is one-to-one on E and $q(E)$ is (weak) ε-Kronecker, then E is (weak) ε-Kronecker.
(2) Suppose $E \subset \Lambda$. Then E is (weak) ε-Kronecker as a subset of Γ if and only if E is (weak) ε-Kronecker as a subset of Λ.

Proof. (1) Suppose $q : \Gamma \to \Gamma/\Lambda$ is one-to-one on E and $q(E)$ is ε-Kronecker. We will show that E is ε-Kronecker. Let $\varphi : E \to \mathbb{T}$ be a function. Because q is one-to-one on E, for each $\gamma, \beta \in E$, if $\beta \neq \gamma$, then $\gamma - \beta \notin \Lambda$. Thus, we can define $\varphi' : q(E) \to \mathbb{T}$ via $\varphi'(\gamma + \Lambda) = \varphi(\gamma)$ for $\gamma \in E$. Since $q(E)$ is ε-Kronecker, there exists $x \in \Lambda^{\perp} = \widehat{\Gamma/\Lambda}$ such that $|\varphi'(\gamma + \Lambda) - x(\gamma + \Lambda)| < \varepsilon$ for all $\gamma \in E$. As $x \in \Lambda^{\perp}$, $|\varphi(\gamma) - \gamma(x)| < \varepsilon$ for all $\gamma \in E$. This means E is ε-Kronecker.

(2) We first suppose E is an ε-Kronecker subset of Γ. Let $\varphi : E \to \mathbb{T}$ be a function. There exists $x \in G$ such that $|\varphi(\gamma) - \gamma(x)| < \varepsilon$ for all $\gamma \in E$. Let $x + \Lambda^{\perp} \in G/\Lambda^{\perp} = \widehat{\Lambda}$. Since $E \subset \Lambda$, for all $\gamma \in E$ we have

$$|\varphi(\gamma) - (x + \Lambda^{\perp})(\gamma)| = |\varphi(\gamma) - \gamma(x)| < \varepsilon.$$

This means E is ε-Kronecker as a subset of Λ. The proof of the converse part of (2) is similar. $\square$

Before we state the next lemma, we introduce the notion of independent sets.

Definition 3.5. Let $E \subset \Gamma$ be a subset without 1. E is independent if whenever $N \in \mathbb{N}$, $\gamma_1, ..., \gamma_N \in E$ and $k_1, ..., k_N \in \mathbb{Z}$ with $\prod_{i=1}^{N} \gamma_i^{k_i} = 1$, then each $\gamma_i^{k_i} = 1$ for all $1 \leq i \leq N$.

Lemma 3.6. *Suppose E is infinite, tor-large and generates Γ. There exists $F \subset \Gamma$ such that the image, $q_F(E)$, has the same cardinality as E and $\Gamma/\langle F\rangle$ is a torsion group.*

Proof. Let $\mathcal{S}$ be the collection of all independent subsets of Γ containing only elements of infinite order. We partially order $\mathcal{S}$ by inclusion. Since a set $A \subset \Gamma$ is independent if and only if every finite subset of A is independent, whenever $(C_\alpha)_\alpha \subset \mathcal{S}$ is a chain, $\bigcup_\alpha C_\alpha$ is still independent and has only elements of infinite order. This means every chain has an upper bound and Zorn's Lemma gives a maximal element in $\mathcal{S}$, denoted by F.

We note that $q_0 : \Gamma \to \Gamma/\Gamma_0$ is one-to-one on $\langle F\rangle$. This is because if we have $\prod_{i=1}^N \gamma_i^{k_i} \in \Gamma_0$ for some $N \geq 1$, $0 \neq k_1, ..., k_N \in \mathbb{Z}$ and $\gamma_1, ..., \gamma_N \in F$, then for some $m \in \mathbb{N}$, we would have $\prod_{i=1}^N \gamma_i^{mk_i} = 1$. By independence, this means each $\gamma_i^{mk_i} = 1$, which contradicts that elements in F have infinite order. For a similar reason as above, we have that each element other than the identity in $\langle F\rangle$ has infinite order.

We will use this fact to show that $|q_F(E)| = |E|$. First, if $q_0(E)$ is finite, then we let $q_0(E) = \{[\lambda_i] : \lambda_i \in E, 1 \leq i \leq n\}$, $n \in \mathbb{N}$, be the coset representatives and write $E = \bigcup_{i=1}^n E_i$, where each E_i is the portion of the coset in E corresponding to λ_i. At least one of the E_i's has cardinality $|E|$, say E_1. We note that q_F is one-to-one on E_1. Indeed, suppose for some $s_1, s_2 \in E_1$, we have $q_F(s_1) = q_F(s_2)$. Write $s_1 = \lambda_1 y_1$ and $s_2 = \lambda_1 y_2$ for some $y_1, y_2 \in \Gamma_0$ and hence of finite order. Now, $q_F(s_1) = q_F(s_2)$ means $s_1 s_2^{-1} \in \langle F\rangle$. But $s_1 s_2^{-1} = y_1 y_2^{-1}$ has finite order and therefore $s_1 s_2^{-1} = 1$ which implies q_F is one-to-one on E_1. Hence, $|q_F(E)| \geq |q_F(E_1)| = |E_1| = |E|$.

Otherwise, $q_0(E)$ is infinite. We claim that $|\langle F\rangle| < |E|$. Because E generates Γ and q_0 is one-to-one on $\langle F\rangle$, $|q_0(E)| = |\langle q_0(E)\rangle| = |q_0(\Gamma)| \geq |q_0(\langle F\rangle)| = |\langle F\rangle|$. If $|\langle F\rangle| \geq |E|$, then $|q_0(E)| = |E|$ and this is a contradiction since E is assumed to be tor-large. This proves the claim.

Let $q_F(E) = \{[\beta_i] : i \in I\}$ with $\beta_i \in E$ the coset representatives. We define

$$f : E \to q_F(E) \times \langle F \rangle$$

by $f(\gamma) = ([\beta_i], \gamma\beta_i^{-1})$, where $[\beta_i] = q_F(\gamma) \in q_F(E)$ for some $i \in I$. This is an injection because for $\gamma, \tau \in E$, if $([\beta_i], \gamma\beta_i^{-1}) = f(\gamma) = f(\tau) = ([\beta_j], \tau\beta_j^{-1})$, then clearly $i = j$ and therefore $\gamma = \tau$.

Hence, $|E| \leq |q_F(E)||\langle F \rangle|$. But $|\langle F \rangle| < |E|$ implies $|q_F(E)| \geq |E|$ and hence we still have $|q_F(E)| = |E|$.

Finally, if there exists an element $\gamma + \langle F \rangle \in \Gamma/\langle F \rangle$ such that $\gamma + \langle F \rangle$ has infinite order, then γ has infinite order and $\gamma^n \notin \langle F \rangle$ for all $n \in \mathbb{N}$. Then we can form the set $F \cup \{\gamma\}$, which is still independent. This contradicts the maximality of F. Thus, $\Gamma/\langle F \rangle$ is a torsion group, as desired. $\square$

Remark. These two lemmas above allow us to assume Γ is a torsion group when E is infinite and tor-large.

Proposition 3.7. *Suppose $E \subset \Gamma$ is independent and contains only elements of order at least $N \geq 2$. Then, E is weak $|1 - e^{i\pi/(N+1)}|$-Kronecker.*

Proof. Since E is independent, for each $\gamma \in E$, $\langle \gamma \rangle \cap \langle E \backslash \{\gamma\} \rangle = \{1\}$. This gives $\langle E \rangle = \oplus_{\gamma \in E} \langle \gamma \rangle$. By Lemma 3.4, E is weak $|1 - e^{i\pi/(N+1)}|$-Kronecker as a subset of Γ if and only if E is so as a subset of $\langle E \rangle$. Thus, we may assume $\Gamma = \langle E \rangle = \oplus_{\gamma \in E} \langle \gamma \rangle$ and $G = \widehat{\Gamma} = \prod_{\gamma \in E} \widehat{\langle \gamma \rangle}$.

Let $\varphi : E \to \mathbb{T}$ be a function. For each $\gamma \in E$, either γ has infinite order or has order $n \geq N$, so $\langle \gamma \rangle$ is either $\mathbb{Z}$ or $\mathbb{Z}_n$. If $\langle \gamma \rangle$ is $\mathbb{Z}$, then γ has dense range and therefore γ is onto by continuity and compactness of G. Hence, there exists $x_\gamma \in \widehat{\langle \gamma \rangle}$ such that $\gamma(x_\gamma) = \varphi(\gamma)$. If $\langle \gamma \rangle = \mathbb{Z}_n$ for some $n \geq N$, then there exists $x_\gamma \in \widehat{\langle \gamma \rangle}$ such that $|\gamma(x_\gamma) - \varphi(\gamma)| \leq |1 - e^{i\pi/(N+1)}|$ because the range of γ is the set of nth roots of unity. We form $x \in G$ by $x = \prod_{\gamma \in E} x_\gamma$.

Each $\gamma \in E$ has the form $\gamma = \prod_{\beta \in E} \lambda_\beta$, where $\lambda_\beta = 1$ for all $\beta \neq \gamma$, $\lambda_\gamma = \gamma$ and $\gamma(x) = \gamma(x_\gamma)$. Thus,

$$|\gamma(x) - \varphi(\gamma)| = |\gamma(x_\gamma) - \varphi(\gamma)| \leq |1 - e^{i\pi/(N+1)}|.$$

Hence, E is weak $|1 - e^{i\pi/(N+1)}|$-Kronecker. □

Notation: For a prime number p, we denote by $\mathcal{C}(p^\infty)$ the discrete p-subgroup of $\mathbb{T}$, ie. the discrete group of all p^n roots of unity.

The general structure of an abelian group is given in the following theorem.

Theorem 3.8 ([18]). *Every abelian group Γ is isomorphic to a subgroup of*

$$\bigoplus_\alpha \mathbb{Q}_\alpha \oplus \bigoplus_\beta \mathcal{C}(p_\beta^\infty),$$

where $\mathbb{Q}_\alpha$ are copies of $\mathbb{Q}$ and p_β are prime numbers.

We start with $\mathcal{C}(p^\infty)$ and $\mathbb{Q}$ and give them the discrete topology.

Proposition 3.9. *Let $\varepsilon > 0$ and p be a prime. Each infinite subset of $\mathcal{C}(p^\infty)$ or $\mathbb{Q}$ has an infinite ε-Kronecker subset.*

Proof. Let $\varepsilon > 0$ be given.

First, we suppose $E \subset \Gamma = \mathcal{C}(p^\infty)$ is an infinite subset. For $m \in \mathbb{Z}$, we can define $\chi_m : \mathcal{C}(p^\infty) \to \mathbb{T}$ by $\chi_m(e^{2\pi ik/p^n}) = e^{2\pi imk/p^n}$. Note that each χ_m is a continuous character on $\mathcal{C}(p^\infty)$ and therefore $\chi_m \in \widehat{\Gamma}$. We choose $N \in \mathbb{N}$ large enough that $|1 - e^{2\pi i/p^N}| < \varepsilon/2$. We will pick the subset $F = (\gamma_i)_{i=1}^\infty \subset E$ inductively.

For γ_1, because E is infinite and the subgroup generated by $\{e^{2\pi i/p^h} : h \leq N\}$ is finite, we can choose $n_1 > N$ and $\gamma_1 = e^{2\pi ik_1/p^{n_1}} \in E$, where $1 \leq k_1 < p$. For $j \geq 1$, we similarly choose n_{j+1} to be such that $n_{j+1} - n_j > N$ and $\gamma_j = e^{2\pi ik_j/p^{n_j}} \in E$ for some $1 \leq k_j < p$. Let $F = (\gamma_j)_{j \geq 1}$.

We will show that F is ε-Kronecker. Let $\varphi : F \to \mathbb{T}$ be given. Since k_1 and p^{n_1} are coprime, the set $\{\chi_m(\gamma_1) : m \in \mathbb{Z}\}$ contains all p^{n_1}th roots of unity. As $n_1 > N$, we can choose $m_1 \in \mathbb{Z}$ such that

$$|\chi_{m_1}(\gamma_1) - \varphi(\gamma_1)| \leq |1 - e^{2\pi i/p^{n_1}}| < \varepsilon/2.$$

Similarly, as $n_2 - n_1 > N$, we can choose $m_2 \in \mathbb{Z}$ such that

$$|\chi_{m_1+m_2p^{n_1}}(\gamma_2) - \varphi(\gamma_2)| = |e^{2\pi ik_2m_2/p^{n_2-n_1}} - \varphi(\gamma_2)e^{-2\pi ik_2m_1/p^{n_2}}| < \varepsilon/2.$$

At the same time, we have

$$\begin{aligned}|\chi_{m_1+m_2p^{n_1}}(\gamma_1) - \varphi(\gamma_1)| &= |e^{2\pi ik_1(m_1+m_2p^{n_1})/p^{n_1}} - \varphi(\gamma_1)| \\ &= |\chi_{m_1}(\gamma_1) - \varphi(\gamma_1)| < \varepsilon/2.\end{aligned}$$

Continuing this way, for each $l \geq 1$ we obtain an integer

$$s_l = m_1 + m_2p^{n_1} + ... + m_lp^{n_1}p^{n_2}...p^{n_{l-1}}$$

such that $|\chi_{s_l}(\gamma_i) - \varphi(\gamma_i)| < \varepsilon/2$ for all $i \leq l$.

Since G is compact, we let χ be a cluster point of the set $(\chi_{s_l})_{l\geq 1}$. For each i and any $\varepsilon' > 0$, we may choose $l' > i$ large enough that $|\gamma_i(s_{l'}) - \gamma_i(\chi)| < \varepsilon'$. Then,

$$|\gamma_i(\chi) - \varphi(\gamma_i)| \leq |\gamma_i(s_{l'}) - \gamma_i(\chi)| + |\gamma_i(s_{l'}) - \varphi(\gamma_i)| < \varepsilon/2 + \varepsilon'.$$

This means $|\gamma_i(\chi) - \varphi(\gamma_i)| \leq \varepsilon/2$ for all i and hence F is ε-Kronecker, as desired.

Now, we consider the case where $\Gamma = \mathbb{Q}$. Similar to the proof of Proposition 2.8, we have that for every Hadamard set $M = (\gamma_j)_{j\geq 1} \subset \mathbb{R}$ of ratio q and every $\varphi : M \to \mathbb{T}$, there exists $\theta \in \mathbb{R}$ such that

$$|\varphi(\gamma_j) - e^{i\gamma_j\theta}| \leq |1 - e^{i\pi/(q-1)}|.$$

Note that $\chi_\theta : \gamma_j \to e^{i\gamma_j\theta}$ is a character on $\mathbb{R}$ (or $\mathbb{Q}$) and therefore is a continuous character on $\mathbb{R}$ (or $\mathbb{Q}$) with the discrete topology. If we choose q large enough such that $|1 - e^{i\pi/(q-1)}| < \varepsilon/3$, then every Hadamard set $(\gamma_i)_{i\geq 1}$ with ratio at least q will be $\varepsilon/3$-Kronecker.

If $E \subset \mathbb{Q}$ is unbounded, we can find a subset $(\gamma_i)_{i\geq 1}$ in E such that $\gamma_{i+1}/\gamma_i \geq q$. Then, $(\gamma_i)_{i\geq 1}$ is ε-Kronecker and we are done.

If $E \subset \mathbb{Q}$ is bounded, we may find a subset $(\gamma_i)_{i\geq 1}$ such that $\gamma_i \to r$ when $i \to \infty$, for some $r \in \mathbb{R}$. (This convergence is respect to the usual topology on $\mathbb{R}$.) By passing to a subsequence, if necessary, we may assume $(\gamma_{j+1}-r)/(\gamma_j - r) < 1/q$ for all $j \geq 1$. We form $E' = \{\gamma_i - r : i \geq 1\}$. Each finite subset of E' is Hadamard with ratio q and therefore is $\varepsilon/3$-Kronecker. By Proposition 2.9, this implies E' is weak $\varepsilon/3$-Kronecker and hence is $\varepsilon/2$-Kronecker. By Corollary 2.20, there is a finite subset $F \subset E'$ such that $(E' \backslash F) + r$ is ε-Kronecker and $(E'\backslash F) + r \subset E \subset \mathbb{Q}$. $\square$

Notation: For each $\gamma \in \bigoplus_{\beta\in B} \mathcal{C}(p_\beta^\infty)$, we define $\pi_\beta(\gamma)$ to be the projection of γ on the β-coordinate. For $x \in \prod_{\beta\in B} \widehat{\mathcal{C}(p_\beta^\infty)}$, we let $\pi_\beta(x)$ be the projection of x on the β-coordinate. We let $B(\gamma) = \{\beta \in B : \pi_\beta(\gamma) \neq 1\}$. For $\gamma \in \bigoplus_{\beta\in B} \mathcal{C}(p_\beta^\infty)$, $B(\gamma)$ is finite.

We give a key lemma.

Lemma 3.10. *Let $\Gamma \subset \bigoplus_{\beta\in B} \Gamma_\beta$, where Γ_β is either $\mathcal{C}(p^\infty)$ for some prime p or $\mathbb{Q}$, and B is infinite. Suppose $N > 1$ and $E \subset \Gamma$ is infinite such that for every $\beta \in B$ there exists $\gamma \in E$ such that $\pi_\beta(\gamma)$ has order at least N. There exists a weak $|1 - e^{\pi i/N}|$-Kronecker subset $F \subset E$ with $|F| = |E|$.*

Proof. First, we observe that $|B| = |E|$. Indeed, by our assumption and the axiom of choice, there is a function $f : B \to E$ such that for each $\beta \in B$, $f(\beta) \in E$ is an element with $\pi_\beta(f(\beta))$ having order at least N. Note that each $\gamma \in E$ has at most finitely many coordinates satisfying this, thus there are at most finitely many β's in B that can be mapped by f to one element in E. This means $|B| \leq |E|\aleph_0 = |E|$. On the other hand, we note that $|E| \leq |\Gamma| \leq |B|\aleph_0 = |B|$. Hence, $|E| = |B|$.

We next construct the set F. If E is countable, we let $\lambda_1 \in E$ and $\beta(1) \in B$ be such that $\pi_{\beta(1)}(\lambda_1)$ has order at least N. Suppose for $i > 1$, $i \in \mathbb{N}$, and we have found $\lambda_j \in E$ for all $1 \leq j < i$ satisfying $B(\lambda_j) \not\subset \bigcup_{j'<j} B(\lambda_{j'})$. Since each $B(\lambda)$ is finite, $\bigcup_{j<i} B(\lambda_j)$ is finite and therefore we may choose $\beta(i) \in B \backslash (\bigcup_{j<i} B(\lambda_j))$ and $\lambda_i \in E$ such that $\pi_{\beta(i)}(\lambda_i)$ has order at least N. Let $F = (\lambda_i)_{i \geq 1}$.

For the case where E is uncountable, we let I be a well-ordered index set of ordinals of cardinality $|B|$ with $1, 2, ...$ as the first elements of I. Let $\lambda_1 \in E$ and $\beta(1) \in B$ be such that $\pi_{\beta(1)}(\lambda_1)$ has order at least N. Suppose for $i > 1$ and we have found $\lambda_j \in E$ for all $1 \leq j < i$ satisfying $B(\lambda_j) \not\subset \bigcup_{j'<j} B(\lambda_{j'})$. If we already have $|\{\lambda_j : 1 \leq j < i\}| = |E|$, we stop. Otherwise, since $B(\gamma)$ is finite for all $\gamma \in E$, if $i < \infty$, then $\bigcup_{j<i} B(\lambda_j)$ is finite, while if i is not finite, then $|\bigcup_{j<i} B(\lambda_j)| = |\{\lambda_j : 1 \leq j < i\}| < |E|$. In either case, $|\bigcup_{j<i} B(\lambda_j)| < |B|$. Thus, may pick $\beta(i) \in B \backslash (\bigcup_{j<i} B(\lambda_j))$ and $\lambda_i \in E$ such that $\pi_{\beta(i)}(\lambda_i)$ has order at least N. Again, we let $F = \{\lambda_i : i \in I\}$ and $|F| = |B| = |E|$.

We now show that F is indeed weak $|1 - e^{\pi i/N}|$-Kronecker. We use (transfinite) induction. By Lemma 3.4, we may assume that $\Gamma = \bigoplus_{\beta \in B} \Gamma_\beta$ and $G = \widehat{\Gamma} = \prod_{\beta \in B} G_\beta$, where $G_\beta = \widehat{\Gamma_\beta}$.

Let $\varphi : F \to \mathbb{T}$ be a function and put $\varepsilon = |1 - e^{i\pi/N}|$.

We can choose $x_1 \in G_{\beta(1)}$ such that $|\varphi(\lambda_1) - \lambda_1(x_1)| \leq \varepsilon$ since $\pi_{\beta(1)}(\lambda_1)$ has order at least N. Suppose $i > 1$ and we have chosen $x_j \in \prod_{j' \leq j} G_{\beta(j')}$ for all $1 \leq j < i$ satisfying $|\varphi(\lambda_{j'}) - \lambda_{j'}(x_j)| \leq \varepsilon$ and $\pi_{\beta(j')}(x_{j'}) = \pi_{\beta(j')}(x_j)$ for all $1 \leq j' \leq j$.

If i has an immediate predecessor i', then we choose $x \in G_{\beta(i)}$ with $|\varphi(\lambda_i) - \lambda_i(x_{i'}x)| \leq \varepsilon$ and just concatenate x at the end of $x_{i'}$ to set $x_i = x_{i'}x$.

For a limit ordinal i, we use a limit argument to find x_i. First, we will show that if i is a limit ordinal, then $(x_j)_{j<i}$ is a Cauchy net in $\prod_{j<i} G_{\beta(j)}$ (add e's to the remaining coordinates of x_j to make $x_j \in \prod_{j<i} G_{\beta(j)}$). Indeed, any open set in $\prod_{j<i} G_{\beta(j)}$ is of the form $\prod_{j<i} U_j$, where U_j are open and $U_j = G_{\beta(j)}$

for all but finitely many j. We let $s < i$ be the largest (finite) ordinal such that $U_s \neq G_{\beta(s)}$. By our construction, for all $k, l > s$ we have that x_k and x_l coincide on all the coordinates j where $U_j \neq G_{\beta(j)}$ and therefore $x_k x_l^{-1} \in \prod_{j<i} U_j$. Hence, $(x_j)_{j<i}$ is a Cauchy net. We let $x_0 = \lim_{j<i} x_j$. Then $x_0 \in \prod_{j<i} G_{\beta(j)}$ satisfies $|\varphi(\lambda_j) - \lambda_j(x_0)| \leq \varepsilon$ and $\pi_{\beta(j)}(x_j) = \pi_{\beta(j)}(x_0)$ for all $j < i$. Now, we choose $x \in G_{\beta(i)}$ such that $|\varphi(\lambda_i) - \lambda_i(x_0 x)| \leq \varepsilon$ and just set $x_i = x_0 x \in \prod_{j \leq i} G_{\beta(j)}$.

Finally, we let $z = \lim_{i \geq 1} x_i$, which exists by similar reasoning to above, and we have $|\varphi(\lambda_i) - \lambda_i(z)| \leq \varepsilon$ for all $i \geq 1$. Hence, F is weak ε-Kronecker. $\square$

3.3 Proof of the existence theorem

We now prove Theorem 3.2.

Proof of Theorem 3.2. (1) Let $\varepsilon > 0$ be given and choose $N > 1$ such that $|1 - e^{i\pi/N}| = \varepsilon' < \varepsilon$.

By Theorem 3.8 and Lemma 3.4, we may assume $\Gamma = \oplus_{\beta \in B} \Gamma_\beta$, where Γ_β is either $\mathcal{C}(p^\infty)$ for some prime p or $\mathbb{Q}$ and B is an index set.

We first suppose E is countable. Consider the collection

$$I = \{i \in B : \exists \gamma \in E, \pi_i(\gamma) \text{ has order at least } N\}.$$

If I is infinite, we appeal to Lemma 3.10.

Otherwise, I is a finite collection. For $i \in I$, we let $Q_i : \Gamma \to \Gamma_i$ be the quotient map. We claim that there exists $j \in I$ such that $Q_j(E)$ is infinite. Indeed, if $Q_i(E)$ are finite for all $i \in I$, because I is finite, we let

$$K = \max\{\text{order}(Q_i(\gamma)) : i \in I, \gamma \in E, \text{order}(Q_i(\gamma)) < \infty\}.$$

Then $q_{K!}(E)$ is finite, which contradicts that E is $K!$-small.

Thus, we may find $j \in I$ such that $Q_j(E)$ is infinite. By Theorem 3.9, there exists an infinite ε-Kronecker set $F' \subset Q_j(E)$. We use the axiom of choice to find $F \subset E$ such that Q_j is one-to-one on F and $Q_j(F) = F'$. By Lemma 3.4, $F \subset E$ is an infinite ε-Kronecker set.

We next suppose E is uncountable. We form the set

$$J := \{\beta \in B : \exists \gamma \in E, \pi_\beta(\gamma) \text{ has order at least } N\}.$$

For $I \subset B$, Q_I denotes the quotient map

$$Q_I : \bigoplus_{\beta \in B} \mathcal{C}(p_\beta^\infty) \to \bigoplus_{\beta \in I} \mathcal{C}(p_\beta^\infty).$$

We claim that $|J| = |E|$. If $|J| < |E|$, we would have

$$|q_{N!}(E)| \leq |q_{N!}(Q_J(E))| \times |q_{N!}(Q_{B \setminus J}(E))|.$$

From our construction, $q_{N!}(Q_{B \setminus J}(E))$ is trivial. But

$$|q_{N!}(Q_J(E))| \leq |Q_J(E)| \leq |J|\aleph_0 < |E|.$$

Hence, $|q_{N!}(E)| < |E|$, which is a contradiction because E is $N!$-small.

Hence, we have $|J| = |E|$ and we appeal to Lemma 3.10.

(2) Since E is N-large, it is tor-large. From Lemma 3.6, there exists a set $F \subset \Gamma$ such that $|q_F(E)| = |E|$ and $\Gamma/\langle F\rangle$ is a torsion group. This means there exists a subset $E' \subset E$ with $|E'| = |E|$ and q_F is one-to-one on E'. By Lemma 3.4, it suffices to show $q_F(E')$ admits an ε-Kronecker subset of the same cardinality.

We may assume Γ is a torsion group and therefore $\Gamma \subset \bigoplus_{\beta \in B} \mathcal{C}(p_\beta^\infty)$ for some primes p_β and index set B. Lemma 3.4 also tells $E \subset \Gamma$ admits an ε-Kronecker subset of the same cardinality if and only if $E \subset \langle E\rangle$ contains such a subset. Hence, we may futher assume $\Gamma = \langle E\rangle \subset \bigoplus_{\beta \in B} \mathcal{C}(p_\beta^\infty)$.

We write $N = p_1^{k_1} p_2^{k_2} ... p_n^{k_n}$ for distinct primes p_i, $1 \leq i \leq n$, $k_i \in \mathbb{N}$ and $n \in \mathbb{N}$. Let $K = p_1^{k_1}$, $K' = p_1^{k_1 - 1}$ and $J = p_2^{k_2} ... p_n^{k_n}$. We will show there exists $F \subset E$ such that $|F| = |E|$ and F is weak ε-Kronecker with $\varepsilon = |1 - e^{i\pi/K}|$.

Since N is the smallest integer such that E is N-large, E is J-small. Hence, $|q_J(E)| = |E|$ and $|q_N(E)| = |q_K(q_J(E))| < |E|$. Thus, by Lemma 3.4 and replacing E with $q_J(E)$, we may assume $N = K = p_1^{k_1}$.

We consider

$$\begin{aligned} &C := \{\beta \in B : p_\beta = p_1, \exists \gamma \in E, \pi_i(\gamma) \text{ has order at least } K\}, \\ &C' := \{\beta \in B : p_\beta = p_1, \exists \gamma \in E, \pi_i(\gamma) \text{ has order less or equal to } K'\}, \\ &D := B \backslash (C \cup C'). \end{aligned}$$

Let $Q_C, Q_{C'}$ and Q_D be the quotient maps given in the proof of (1). Since $E \subset Q_C(E) \times Q_{C'}(E) \times Q_D(E)$, we have

$$|q_{K'}(E)| \leq |q_{K'}(Q_C(E))| \times |q_{K'}(Q_{C'}(E))| \times |q_{K'}(Q_D(E))|. \tag{$*$}$$

By our construction, $q_{K'}(Q_{C'}(E))$ is trivial. Since E is K-large,

$$|q_{K'}(Q_D(E))| = |Q_D(E)| \leq |q_K(E)| < |E|. \tag{$**$}$$

If E is uncountable, we claim that $|C| = |E|$. Suppose, otherwise, that $|C| < |E|$. We have

$$|q_{K'}(Q_C(E))| \leq |Q_C(E)| \leq |C| \aleph_0 < |E|.$$

Hence, $(*)$ gives $|q_{K'}(E)| < |E|$, which is a contradiction because E is K'-small. Thus, $|C| = |E|$ and we appeal to Lemma 3.10.

Finally, assume E is countably infinite. If C is infinite, then $|C| = |E|$ and we appeal to Lemma 3.10. Otherwise C is finite. Since $|q_{K'}(Q_D(E))| < |E|$ by (**),

$|q_{K'}(Q_D(E))|$ is finite. Moreover, since E is K'-small, $|q_{K'}(E)| = |E|$ and $(*)$ implies $q_{K'}(Q_C(E))$ is infinite and therefore $Q_C(E)$ is infinite. Since C is finite, there exists $c \in C$ such that $Q_c(E)$ is an infinite subset of $C(p_1^\infty)$ and then we appeal to Proposition 3.9. $\square$

Remark. We note that the result of Theorem 3.2 (2) is the best possible. Consider $G = \mathbb{Z}_p^{\mathbb{N}}$ and for $j \geq 1$, let $\gamma_j \in \Gamma = \bigoplus_1^\infty \mathbb{Z}_p$ have $e^{2\pi i/p}$ in coordinate j and 1 in the other coordinates. Let $E = (\gamma_j)_{j \geq 1}$. Let $\varphi : E \to \mathbb{T}$ be given by $\varphi(\gamma_j) = e^{\pi i/p}$ for all $j \geq 1$. Since $\gamma_i(x) \in \{e^{2k\pi i/p} : 0 \leq k \leq p-1\}$ for all $x \in G$, the best interpolation error is $|1 - e^{\pi i/p}|$.

Chapter 4

Interpolation sets revisited

In this chapter, we will show an alternate and more topological approach to the result that every infinite subset of the dual of a compact, abelian and locally connected group G contains an infinite ε-Kronecker subset.

Definition 4.1. G is locally connected if for every $x \in G$ and open set $V \subset G$ containing x, there exists a connected open set U such that $x \in U \subset V$.

Remark. Note that not every locally connected compact abelian group is connected. Consider finite abelian discrete groups. They are locally connected but not connected.
From [12] (Proposition 4.6), we also have that not every connected compact abelian group is locally connected.

Lemma 4.2. *Let $E \subset \Gamma$ be infinite. For all neighbourhoods U of $e \in G$, we have $\bigcup_{\gamma \in E} \gamma(U) = \mathbb{T}$.*

Proof. Suppose otherwise that there exists a neighbourhood V of $e \in G$ and $p \in \mathbb{T}$ such that $p \notin \bigcup_{\gamma \in E} \gamma(V)$. Note that $p \neq 1 \in \mathbb{T}$. Let U be a neighbourhood of $1 \in \mathbb{T}$.

We will first show that there exists a neighbourhood W of $e \in G$ such that $\gamma(W) \subset U$ for all $\gamma \in E$. Choose $n \in \mathbb{N}$ large enough that the connected component of $\mathbb{T} \backslash \{x \in \mathbb{T} : x^n = p\}$ containing 1, $C_n(1)$, satisfies $C_n(1) \subset U$. Since the map $f_n : G \to G$ given by $f_n(w) = w^n$ is continuous, there exists a neighbourhood W

of $e \in G$ such that $f_n(W) \subset V$. Since G is locally connected, we may assume W is connected. For all $\gamma \in E$, we have

$$\gamma(f_n(W)) \subset \gamma(V) \subset \mathbb{T}\backslash\{p\}.$$

Then, for all $\gamma \in E$ and $w \in W$, we have $\gamma(f_n(w)) = \gamma(w^n) = \gamma(w)^n \in \mathbb{T}\backslash\{p\}$. Hence, for all $\gamma \in E$, $1 \in \gamma(W) \subset \mathbb{T}\backslash\{x \in \mathbb{T} : x^n = p\}$. Since W is connected and $\gamma \in E$ is continuous, $\gamma(W)$ is also connected. Hence, $\gamma(W) \subset C_n(1) \subset U$ for all $\gamma \in E$.

That proves E is equicontinuous. By viewing $E \subset (C(G), ||\cdot||_\infty)$, the Arzela-Ascoli Theorem implies there are cluster points of E. This implies E has cluster points in $L^2(G)$. But $E \subset \Gamma$ is orthonormal and therefore E does not have any cluster points in $L^2(G)$. This is a contradiction. □

Theorem 4.3 ([2]). *If $E \subset \Gamma$ is infinite, then for any sequence $(\lambda_k)_{k=1}^\infty$ of positive real numbers there is an infinite subset $F = (\gamma_k)_{k=1}^\infty \subset E$ such that for any sequence $(I_k)_{k=1}^\infty$ of intervals in $\mathbb{T}$ with length $l(I_k) = \lambda_k$, there exists $x \in G$ such that $\gamma_k(x) \in I_k$ for all $k \geq 1$.*

Proof. We first inductively construct a countable family of neighbourhoods, $(K_k)_{k=1}^\infty$, of $e \in G$ and a sequence $(\gamma_k)_{k=1}^\infty \subset E$ such that $(K_k)_{k=1}^\infty$ are connected, compact and symmetric, $\gamma_k(K_k) = \mathbb{T}$, $l(\gamma_k(2K_{k+1})) < \lambda_k$ and $2K_{k+1} \subset K_k$ for all $k \geq 1$.

Let C be the connected component of $e \in G$. Since G is locally connected, C is open. From Lemma 4.2, we let $\gamma_1 \in E$ be such that $-1 \in \gamma_1(C)$. Since C is a connected, open and therefore closed subgroup of G and γ_1 is a continuous homomorphism, $\gamma_1(C) \subset \mathbb{T}$ is a non-trivial connected subgroup. But subgroups of $\mathbb{T}$ are either finite or dense, hence since $\gamma_1(C)$ is also compact, we have that $\gamma_1(C) = \mathbb{T}$ and therefore we can choose $K_1 = C$.

Suppose for some $k \geq 1$, compact neighbourhoods of $e \in G$, $\{K_1, ..., K_k\}$, and $\{\gamma_1, ..., \gamma_k\} \subset E$ are found. We choose K_{k+1} be a neighbourhood of $e \in G$

such that K_{k+1} is compact, connected and symmetric with $l(\gamma_k(2K_{k+1})) < \lambda_k$ and $2K_{k+1} \subset K_k$. From Lemma 4.2, $\bigcup_{\gamma \in E} \gamma(K_{k+1}) = \mathbb{T}$. We choose $\gamma_{k+1} \in E$ such that $-1 \in \gamma_{k+1}(K_{k+1})$. Since γ_{k+1} is a continuous character and K_{k+1} is connected and symmetric, $\gamma_{k+1}(K_{k+1}) \subset \mathbb{T}$ is connected and symmetric. But since $1, -1 \in \gamma_{k+1}(K_{k+1})$, it follows that $\gamma_{k+1}(K_{k+1}) = \mathbb{T}$ and the inductive construction of $(K_k)_{k\geq 1}$ and $(\gamma_k)_{k\geq 1}$ is complete.

Let $(I_k)_{k\geq 1}$ be a sequence of intervals in $\mathbb{T}$ such that $l(I_k) = \lambda_k$. Let t_k be the middle points of I_k for $k \geq 1$. We inductively construct a sequence $(x_k)_{k\geq 1} \subset G$ with $x_k \in K_k$ such that $\gamma_k(x_1 + ... + x_k) = t_k$ and $\gamma_k(x_1 + ... + x_k + 2K_{k+1}) \subset I_k$ for all $k \geq 1$.

For $k = 1$, we choose $x_1 \in C = K_1$ such that $\gamma_1(x_1) = t_1$. Suppose for some $k \geq 1$, $x_1, ..., x_k \in G$ are found as required. Since $\gamma_{k+1}(K_{k+1}) = \mathbb{T}$, we have that

$$\gamma_{k+1}(x_1 + ... + x_k + K_{k+1}) = \gamma_{k+1}(x_1 + ... + x_k)\gamma_{k+1}(K_{k+1}) = \mathbb{T}.$$

Hence, we can choose $x_{k+1} \in K_{k+1}$ such that $\gamma_{k+1}(x_1 + ... + x_k + x_{k+1}) = t_{k+1}$. We have

$$\begin{aligned}\gamma_{k+1}(x_1 + ... + x_{k+1} + 2K_{k+2}) &= \gamma_{k+1}(x_1 + ... + x_{k+1})\gamma_{k+1}(2K_{k+2}) \\ &= t_{k+1}\gamma_{k+1}(2K_{k+2}) \subset I_{k+1},\end{aligned}$$

since K_{k+2} is symmetric and $l(\gamma_{k+1}(2K_{k+2})) < \lambda_{k+1}$. This completes the induction step.

Finally, we form $C_k = x_1 + ... + x_k + 2K_{k+1}$ for $k \geq 1$. Each C_k is compact and $\gamma_k(C_k) \subset I_k$. Moreover, since $2K_{k+2} \subset K_{k+1}$, $x_{k+1} + 2K_{k+2} \subset 2K_{k+1}$ and therefore $C_k \supset C_{k+1}$. Since G is compact and each C_k is compact, by the finite intersection property, we have $\bigcap_{k\geq 1} C_k \neq \emptyset$ and we choose $x \in \bigcap_{k\geq 1} C_k \neq \emptyset$. We have $\gamma_k(x) \in I_k$ for all $k \geq 1$, as desired. $\square$

Corollary 4.4. *If $E \subset \Gamma$ is infinite, then there exists an infinite subset $F \subset E$ such that given any $\varepsilon > 0$, there exists a finite subset $F_\varepsilon \subset F$ such that $F \backslash F_\varepsilon$ is*

ε-*Kronecker.*

Proof. We choose $\lambda_k = 1/k$ for $k \geq 1$ and find $F = (\gamma_k)_{k\geq 1}$ from Theorem 4.3. For any given $\varepsilon > 0$, we find $N \in \mathbb{N}$ large enough that $1/N < \varepsilon$. We let $F_\varepsilon = (\gamma_k)_{k\leq N}$ and $\varphi : F\backslash F_\varepsilon \to \mathbb{T}$. For each $k \geq N$, we let I_k be the interval in $\mathbb{T}$ with length λ_k centered at $\varphi(\gamma_k)$. From Theorem 4.3, there exists $x \in G$ such that $\gamma_k(x) \in I_k$ for all k. Hence, $|\varphi(\gamma_k) - \gamma_k(x)| < 1/(2k) < 1/N < \varepsilon$ for all $k \geq N$, which implies $F\backslash F_\varepsilon$ is ε-Kronecker. $\square$

Remark. 1. Theorem 4.3 is actually a special case of a much more abstract theorem proven in [2]. This work of Galindo and Hernandez was motivated by [1], and was extended further in [3] where the authors use topological ideas to study the more general problem of the existence of I_0 sets in locally compact MAP groups. (We call topological groups, Γ, MAP (maximally almost periodic) if the finite dimensional representations of Γ separate points in Γ.)

2. The existence of large I_0 sets in duals of compact non-abelian groups was also investigated in [9] and [6].

Bibliography

[1] E. K. van Douwen, *The maximal totally bounded group topology on G and the biggest minimal G-space, for abelian groups G*, Topology Appl., 34(1):69-91, 1990.

[2] J. Galindo and S. Hernandez, *The concept of boundedness and the Bohr compactification of a MAP Abelian group*, Fund. Math. 159: 195-218, 1999.

[3] J. Galindo and S. Hernandez, *Interpolation sets and the Bohr topology of locally compact groups*, Adv. Math. 188: 51-68, 2004.

[4] C. C. Graham and K. E. Hare, *ε-Kronecker and I_0 sets in abelian groups, III: interpolation by measures on small sets*, Studia Math. 171: 15-32, 2005.

[5] C. C. Graham and K. E. Hare, *ε-Kronecker and I_0 sets in abelian groups, IV: interpolation by non-negative measures*, Studia Math. 177: 9-24, 2006.

[6] C. C. Graham and K. E. Hare, *I_0 sets for compact, connected groups: interpolation with measures that are nonnegative or of small support*, J. Austral. Math. Soc., 84(2):199-225, 2008.

[7] C. C. Graham and K. E. Hare, *Existence of Large ε-Kronecker and FZI_0 sets in discrete abelian groups*, Colloquium Math.,127(1):1-15, 2012.

[8] C. C. Graham and K. E. Hare, *Interpolation and Sidon Sets for Compact Groups*, Springer, New York, 2013.

[9] K. E. Hare and L. T. Ramsey, *I_0 sets in non-abelian groups*, Math. Proc. Cambridge Philos. Soc., 135:81-98, 2003.

[10] S. Hartman, *On interpolation by almost periodic functions*, Colloquium Math, 8:99-101, 1961.

[11] S. Hartman and C. Ryll-Nardzewski, *Almost periodic extensions of functions*, Colloquium Math., 12:23–39, 1964.

[12] K. H. Hofmann and S. A. Morris, *The Lie Theory of Connected Pro-Lie Groups-A Structure Theory for Pro-Lie Algebras, Pro-Lie Groups and Connected Locally Compact Groups*, EMS Publishing House, Zurich, 2006.

[13] N. J. Kalton. *On vector-valued inequalities for Sidon sets and sets of interpolation*, Colloquium Math., 64(2):233-244, 1993.

[14] K. Kunen and W. Rudin, *Lacunarity and the Bohr topology*, Math. Proc. Cambridge Philos. Soc., 126:117-137, 1999.

[15] J. F. Mela, *Sur les ensembles d'interpolation de C. Ryll-Narzewski et de S. Hartman*, Studia Math., 29:168-193, 1968.

[16] L. T. Ramsey, *A theorem of C. Ryll-Nardzewski and metrizable l.c.a. groups*, Proc. Amer. Math. Soc., 78(2):221-224, 1980.

[17] L. T. Ramsey, *Comparisons of Sidon and I_0 sets*, Colloquium Math., 70(1):103-132, 1996.

[18] J. J. Rotman, *An Introduction to the Theory of Groups*, 4th ed., Grad. Texts in Math. 148, Springer, New York, 1995.

[19] W. Rudin, *Fourier analysis on groups*, Interscience Tracts in Pure and Applied Math., No. 12. Wiley, New York, 1962.

[20] C. Ryll-Nardzewski, *Concerning almost periodic extensions of functions*, Colloquium Math., 12:235-237, 1964.

[21] A. Zygmund, *Trigonometric series*, Cambridge University Press, Cambridge, 1959.